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A LOWER TRIANGULAR HERMITE NORMAL FORM FOR PROJECTION-REGULAR LATTICE RULES*

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Abstract

The structure of lattice rules has been studied using two different approaches. One of them is based on the generator matrix B of the dual of the integration lattice while the other approach is based on the representation of lattice rules in $D-Z$ form. The former approach has previously made the assumption that the Hermite normal form of the matrix B is upper triangular. However, for the special case of projection-regular rules in which the principal projections have the maximum possible number of distinct quadrature points, it is possible to specify a unique upper triangular matrix Z. The corresponding matrix $B = D(Z^T)^{-1}$ is then lower triangular. This leads us to investigate the lower triangular Hermite normal form for projection-regular rules. The results obtained give conditions on the generator matrix which allow projection-regular rules to be easily recognized.

1 Introduction

Lattice rules are equal-weight quadrature rules which may be used to approximate the s-dimensional integral

$$I(f) = \int_{[0,1]^s} f(\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

These lattice rules may (as shown in [11]) always be expressed in the canonical form

$$Q(f) = \frac{1}{d_1 d_2 \cdots d_r} \sum_{i_1=0}^{d_1-1} \sum_{i_2=0}^{d_2-1} \cdots \sum_{i_r=0}^{d_r-1} f\left(\left\{ i_1 \frac{\mathbf{z}_1}{d_1} + i_2 \frac{\mathbf{z}_2}{d_2} + \cdots + i_r \frac{\mathbf{z}_r}{d_r} \right\}\right), \tag{1}$$

where the positive integers d_i, known as the *invariants*, satisfy $d_{i+1} \mid d_i$, $1 \le i < r$ with $d_r > 1$. Moreover, the $\mathbf{z}_i \in \mathbb{Z}^s$ are linearly independent and the braces around a vector indicate that we take the fractional part of each component in the vector. The

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number of distinct quadrature points in a lattice rule is known as the *order* of the rule and for a lattice rule in canonical form is given by $\prod_{i=1}^{r} d_i$.

The number $r \leq s$ is known as the *rank* of the rule. We may obtain a r-cycle $D-Z$ form for such rules by taking $D = \text{diag}\{d_1, d_2, \ldots, d_r\}$ and Z to be the $r \times s$ matrix whose i-th row is $\mathbf{z}_i$. In this paper we shall always assume that the r-cycle $D-Z$ forms arise from canonical forms of lattice rules (see (1)). Sometimes it is convenient to have an extended canonical s-cycle $D-Z$ form in which D and Z are square $s \times s$ matrices. This may be done by including the trivial invariants $d_{r+1} = \cdots = d_s = 1$ and choosing $\mathbf{z}_{r+1}, \ldots, \mathbf{z}_s \in \mathbb{Z}^s$ arbitrarily.

In this paper, we shall consider lattice rules that are known as projection-regular rules. Not all lattice rules are projection-regular. However, the class of projection-regular rules is wide enough to contain many interesting rules. It is interesting to note that Maisonneuve [7] restricted her search for good lattice rules of rank-1 to projection-regular rules by taking the first component of $\mathbf{z}$ to be 1. In order to define projection-regular rules, we start with the projections of a lattice rule. For $1 \leq \ell \leq s$, a ℓ-dimensional projection of a lattice rule is the ℓ-dimensional rule obtained when all of specified $(s-\ell)$ components of each quadrature point are omitted. As a special case, if the last $(s-\ell)$ components are omitted, then the resulting rule will be referred to as the ℓ-dimensional principal projection of the original rule. These principal projections are also lattice rules. An s-dimensional lattice rule Q having invariants $d_1, d_2, \ldots, d_s$, is said to be projection-regular if for $1 \leq \ell \leq s$, the ℓ-dimensional principal projections have order $d_1 d_2 \cdots d_\ell$. In other words, projection-regular lattice rules are those in which all the principal projections have the maximum possible order.

In the $D-Z$ form corresponding to (1), the value of r and the matrix D are unique. However, there remain many possibilities for Z. In certain cases it is possible to specify Z uniquely. For instance, a unique Z is available for projection-regular rules (see [12]) and for prime-power lattice rules, that is, rules whose order is a power of some prime (see [1]). In the latter case, the unique $D-Z$ form developed is known as an ultratriangular form. Associated with each ultratriangular form is a set of column indices. A unique Z is also available for a special class of lattice rules whose prime-power components have ultratriangular forms for which the sets of corresponding column indices satisfy a certain consistency condition (see [8]).

These lattice rules are so called because the quadrature points of these rules are all the points in $[0,1)^s$ that belong to some *integration lattice* Λ, that is, a discrete set of points in $\mathbb{R}^s$ which is closed under normal addition and subtraction and which contains $\mathbb{Z}^s$ as a sublattice. Associated with each integration lattice Λ is the *dual lattice* which comprises all $(h_1, \ldots, h_s) \in \mathbb{Z}^s$ such that

$$h_1 x_1 + \cdots + h_s x_s \in \mathbb{Z}, \quad \forall \mathbf{x} = (x_1, \ldots, x_s) \in \Lambda.$$

The dual lattice plays a very important role in the error analysis of lattice rules (see [10] for further details). It may be specified by an $s \times s$ generator matrix B (see [4]). This matrix B is an integer matrix which may always be written in a unique triangular form by carrying out row operations which do not change the generated dual lattice. This unique form for integer matrices is known in the literature as the Hermite normal form (see for example, [9]) and may either be upper triangular or lower triangular.

All the theory based on this generator matrix of the dual lattice has so far made the assumption that the Hermite normal form is upper triangular. For instance, this upper triangular form has been used previously to count the number of lattice rules, to obtain information about so-called sublattices and superlattices, and to form the basis of a search program for good lattice rules (for example, see [4], [5], and [6]). However, since the unique Z for projection-regular lattice rules is upper triangular, the corresponding $B = D(Z^T)^{-1}$ is lower triangular. This leads us to investigate the lower triangular Hermite normal form for projection-regular rules. This unique representation is given in Section 3. To obtain this result, we make use of their unique $D-Z$ form and so in the next section, results from [12] relating to the unique $D-Z$ form for projection-regular rules are given.

2 Unique $D - Z$ Form for Projection-Regular Lattice Rules

Projection-regular rules, as mentioned earlier, are special classes of lattice rules in which all the principal projections have the maximum possible order. For such rules, as indicated earlier, the $s \times s$ matrix Z in the extended s-cycle $D - Z$ form may be determined uniquely. The following result which gives this unique form is taken from [12].

THEOREM 1. Suppose we have an extended canonical s-cycle $D - Z$ form for a projection-regular rule. Moreover, suppose the matrix Z has the following properties:

(a) $z_{ij} = 0, \quad 1 \le j < i \le s,$

(b) $z_{ii} = 1, \quad 1 \le i \le s,$

(c) $0 \le z_{ij} < \frac{d_i}{d_j}, \quad 1 \le i < j \le s.$

Then such a Z is unique.

By using this unique Z, we may find the number of projection-regular rules having a given set of invariants. Hence, we have the following result (which is stated and proved in [10]).

THEOREM 2. The number of projection-regular lattice rules having invariants $d_1, d_2, \ldots, d_s$ is given by $\quad d_1^{s-1} d_2^{s-3} \cdots d_{s-1}^{3-s}\, d_s^{1-s}$.

3 A Unique Lower Triangular Form for Projection-Regular Lattice Rules

In order to obtain a unique lower triangular representation for the matrix B of projection-regular rules, we shall first define the lower triangular Hermite normal form.

DEFINITION 1. An $s \times s$ integer matrix C is in lower triangular Hermite normal form if and only if

(a) $c_{ii} \geq 1, \quad 1 \leq i \leq s,$

(b) $c_{ij} = 0, \quad 1 \leq i < j \leq s,$

(c) $0 \leq c_{ij} < c_{jj}, \quad$ otherwise.

We may use elementary row operations (see [12]) to transform any given integer matrix into this lower triangular Hermite normal form. After this is done or during the process of doing this, it is straightforward to arrange the subdiagonal elements such that they satisfy condition (c) of the above definition.

THEOREM 3. For every lattice rule, the corresponding dual lattice has a unique generator matrix B in lower triangular Hermite normal form.

PROOF. The result follows from Theorem 4.2 of [9]. Given any generator matrix for the dual of an integration lattice, the dual lattice is unchanged by any of the row operations required to transform it into lower triangular Hermite normal form.

For projection-regular rules, the unique Z-matrix given in Theorem 1 is unimodular since it is upper triangular with all the elements in the diagonal being 1. In order to derive a corresponding unique lower triangular form for the matrix B from this unique $D-Z$ form, we require the next result which is a consequence of Theorem 2.2 of [2].

THEOREM 4. Suppose that the lattice rule is expressed in an extended s-cycle $D-Z$ form with a Z-matrix that is unimodular. Then a generator matrix for the corresponding dual lattice is given by $B = D(Z^T)^{-1}$.

The form of this lower triangular B is given in the following result.

THEOREM 5. For a rank-r projection-regular lattice rule having the unique s-cycle $D-Z$ form given in Theorem 1, the matrix $B = D(Z^T)^{-1}$ is given by

$$b_{ij} = \begin{cases} 0, & j > i \quad \text{or} \quad r < j < i, \\ d_i, & j = i \quad \text{and} \quad 1 \leq i \leq s, \\ d_i \displaystyle\sum_{\mathbf{K} \in S_{ij}} z_{jk_1} z_{k_1 k_2}, \cdots z_{k_\theta i} \times \operatorname{sign}(\mathbf{K}), & j < i \quad \text{and} \quad j \leq r. \end{cases} \tag{2}$$

The elements of the set S_{ij} are generalized integers $\mathbf{K} = (k_1, k_2, \ldots, k_\theta)$ such that

$$j < k_1 < k_2 < \cdots < k_\theta < i.$$

The set S_{ij} is empty when $i = j+1$ and it may contain at most 2^{i-j-1} elements (because $z_{\ell m} = 0$ for $r < \ell < m$, some of the elements vanish). Associated with each $\mathbf{K}$ is $\operatorname{sign}(\mathbf{K}) = (-1)^{\theta+1}$ which takes the value 1 when the number of integers is odd and the value -1 when the number of integers is even including zero.

PROOF. This result follows from arguments similar to those in Appendix E of [3].

To give a better understanding of the form (2) for the matrix B, we now give two examples.

EXAMPLE 1. For a six-dimensional projection-regular rule with rank 2, the matrix $B = D(Z^T)^{-1}$ is given by

$$B = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ -z_{12}d_2 & d_2 & 0 & 0 & 0 & 0 \\ z_{12}z_{23} - z_{13} & -z_{23} & 1 & 0 & 0 & 0 \\ z_{12}z_{24} - z_{14} & -z_{24} & 0 & 1 & 0 & 0 \\ z_{12}z_{25} - z_{15} & -z_{25} & 0 & 0 & 1 & 0 \\ z_{12}z_{26} - z_{16} & -z_{26} & 0 & 0 & 0 & 1 \end{bmatrix}.$$

EXAMPLE 2. For a six-dimensional rank-3 projection-regular lattice rule, the matrix $B = D(Z^T)^{-1}$ is given by

$$B = \begin{bmatrix} d_1 & 0 & 0 & 0 & 0 & 0 \\ -z_{12}d_2 & d_2 & 0 & 0 & 0 & 0 \\ (z_{12}z_{23} - z_{13})d_3 & -z_{23}d_3 & d_3 & 0 & 0 & 0 \\ -z_{12}z_{23}z_{34} + z_{12}z_{24} + z_{13}z_{34} - z_{14} & z_{23}z_{34} - z_{24} & -z_{34} & 1 & 0 & 0 \\ -z_{12}z_{23}z_{35} + z_{12}z_{25} + z_{13}z_{35} - z_{15} & z_{23}z_{35} - z_{25} & -z_{35} & 0 & 1 & 0 \\ -z_{12}z_{23}z_{36} + z_{12}z_{26} + z_{13}z_{36} - z_{16} & z_{23}z_{36} - z_{26} & -z_{36} & 0 & 0 & 1 \end{bmatrix}.$$

Notice that the matrix B given in (2) is lower triangular. This justifies our decision to consider lower triangular representations of B for projection-regular rules. Once we have the matrix B in this form, we may carry out a series of row operations on it such that it becomes a special case of the lower triangular Hermite normal form given in Definition 1. We then have the following result.

THEOREM 6. Let a rank-r projection-regular lattice rule be given in the unique s-cycle $D-Z$ form, as defined in Theorem 1. Then the matrix B given by $B = D(Z^T)^{-1}$ may be expressed uniquely in lower triangular Hermite normal form with elements satisfying

(a) $b_{ii} = d_i, \quad 1 \le i \le s,$

(b) $b_{ij} = 0, \quad 1 \le i < j \le s,$

(c) $0 \le b_{ij} < b_{jj}, \quad 1 \le j < i \le s,$

(d) $b_{ij}/b_{ii} \in \mathbb{Z}$; that is, b_{ij} has a factor $b_{ii} = d_i, \quad j < i \le r.$

PROOF. In order to transform the matrix B given in (2) into this lower triangular Hermite normal form, we may carry out row operations of the form,

$$\mathbf{b}_i' = \mathbf{b}_i + \lambda \mathbf{b}_j, \quad \text{where} \quad \lambda \in \mathbb{Z}, \quad i \ne j. \tag{3}$$

The matrix B given in (2) is already in a lower triangular form with d_i's on the main diagonal. Thus, we only need to make the entries b_{ij} lying below the main diagonal nonnegative and less than b_{jj}. This may be done by using the row operation (3) with $\lambda = -\left\lfloor \frac{b_{ij}}{b_{jj}} \right\rfloor$. In particular, the j-th component of $\mathbf{b}_i'$ is given by

$$b_{ij}' = b_{ij} - \left\lfloor \frac{b_{ij}}{b_{jj}} \right\rfloor b_{jj} = \left(\frac{b_{ij}}{b_{jj}} - \left\lfloor \frac{b_{ij}}{b_{jj}} \right\rfloor \right) b_{jj},$$

which clearly satisfies $0 \le b'_{ij} < b_{jj}$. These row operations must be ordered in such a way that once b_{ij} is changed, it is not altered again. This is achieved if the row operations are carried out in the following order. In (3), for every value of i going from s down to $r+1$ we take j from r down to 1. Then all the elements below the r-th row will satisfy the conditions of the above theorem.

The rest of the entries b_{ij} for $j < i \le r$ must also be less than b_{jj}. For these entries we perform the above row operation by taking for every value of i from r down to 2, the values of j from $i-1$ down to 1. We need to verify that the non-trivial factors d_i are preserved in these entries. To do this, we note that the entries b_{ij} and d_j both have the factor d_i for $j < i \le r$ (this follows from Theorem 5 and the fact that $d_{i+1} \mid d_i$ for $1 \le i < r$, respectively); that is,

$$b_{ij} = \beta_1 d_i, \quad d_j = \beta_2 d_i,$$

where $\beta_1, \beta_2 \in \mathbb{Z}$. It then follows that

$$b'_{ij} = b_{ij} - \left\lfloor \frac{b_{ij}}{b_{jj}} \right\rfloor b_{jj} = d_i \left(\beta_1 - \left\lfloor \frac{\beta_1}{\beta_2} \right\rfloor \beta_2 \right).$$

Hence, the factors d_i are preserved in entries b_{ij} for $j < i \le r$.

We remark that the unique B given in Theorem 6 may be used to obtain the number of projection-regular rules having a given set of invariants. This may be done by first noting that the entries b_{ij} for $j < i \le r$ have a factor d_i. Moreover, entries b_{ij} in the j-th column of B must satisfy $b_{ij} < d_j$. Hence the total number of choices for b_{ij} when $j < i \le r$ is d_j/d_i. The rest of the entries b_{ij} below the diagonal must be less than d_j. By considering each of the columns of this unique B in turn, we see that the total number of possibilities correspond to the number of projection-regular lattice rules, as given in Theorem 2.

We also remark that if we have a matrix B in the form defined by Theorem 6, then it always represents a projection-regular rule with the rank equal to the number of entries on the main diagonal that are greater than one.

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