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SOME BOUNDS FOR THE SPECTRAL RADIUS OF THE HADAMARD PRODUCT OF MATRICES*

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Abstract

Some bounds for the spectral radius of the Hadamard product of two nonnegative matrices are given. Some results involve M-matrices.

1 Introduction

For any two $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$, the Hadamard product of A and B is $A \circ B := (a_{ij}b_{ij})$. It is known [6, p.358] that if $A, B \in \mathbb{R}_{n\times n}$ are nonnegative matrices, then

$$\rho(A \circ B) \leq \rho(A)\rho(B), \tag{1}$$

where $\rho(A)$ denotes the spectral radius of A, and in this case it is equal to the Perron root of A. See [3] for some generalizations. It is a neat inequality and bears some symmetry, i.e., the inequality remains unchanged if A and B are switched (unlike the usual matrix multiplication, Hadamard product is commutative).

Due to the monotonicity of the Perron root of the nonnegative $B \in \mathbb{R}_{n\times n}$ [5, p.491], one has

$$\max_{i=1,\ldots,n} b_{ii} \leq \rho(B). \tag{2}$$

The lower bound is clearly attained when B is nonnegative diagonal. Is it possible to have a better bound like the following for $\rho(A \circ B)$, where $A, B \geq 0$?

$$\rho(A \circ B) \leq \rho(A) \max_{i=1,\ldots,n} b_{ii} \tag{3}$$

If the suspected inequality (3) is true, it would provide a better and computationally simpler upper bound. However the answer is negative in view of the following example.

EXAMPLE 1. Let

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$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad A \circ B = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}.$$

Evidently $\rho(A) = 1$, $\rho(B) = 3$, $\rho(A \circ B) = 2$ and $\max_{i=1,2} b_{ii} = 1$. So

$$\rho(A \circ B) \not\leq \rho(A) \max_{1 \leq i \leq 2} b_{ii}.$$

This example can be extended to the $n \times n$ ($n \geq 3$) case by attaching I_{n-2} via direct sum.

In the next section we will provide a necessary condition for (3) to be valid. It turns out the condition is satisfied by an important class of matrices called inverse M-matrices. In Section 3 we will provide another upper bound.

2 Some Bounds and Diagonal Dominance

A matrix A is said to be *diagonally dominant of its row entries* (respectively, of its columns entries) if

$$|a_{ii}| \geq |a_{ij}| \qquad (\text{respectively} \quad |a_{ii}| \geq |a_{ji}|)$$

for each $i = 1, \ldots, n$ and all $j \neq i$. Similarly we define *diagonally subdominant* of its row entries (respectively, of its columns entries) by reversing the inequalities. Strictly diagonal dominance is defined similarly [6, p.125].

THEOREM 1. Let $A \geq 0$, $B \geq 0$ be $n \times n$ nonnegative matrices. If there exists a positive diagonal D such that DBD^{-1} is diagonally dominant of its column (or row) entries, then

$$\rho(B) \leq \operatorname{tr} B, \tag{4}$$

and

$$\rho(A \circ B) \leq \rho(A) \max_{i=1,\ldots,n} b_{ii}. \tag{5}$$

Similarly, if there exists a positive diagonal D such that DBD^{-1} is diagonally subdominant of its column (or row) entries, then

$$\rho(B) \geq \operatorname{tr} B$$

and

$$\rho(A) \min_{i=1,\ldots,n} b_{ii} \leq \rho(A \circ B).$$

PROOF. Notice that $A \circ (DBD^{-1}) = D(A \circ B)D^{-1}$ and hence $\rho(A \circ B) = \rho(A \circ (DBD^{-1}))$. Moreover the diagonal entries of B and DBD^{-1} are the same. So we may assume that B is diagonally dominant of its column (or row) entries. Suppose B is diagonally dominant of its column entries. Then

$$A \circ B \leq A \,\mathrm{diag}\,(b_{11}, \ldots, b_{nn}) \leq A \max_{i=1,\ldots,n} b_{ii}. \tag{6}$$

By the monotonicity of the Perron root [5, p.491]

$$\rho(A \circ B) \leq \rho(A \,\mathrm{diag}\,(b_{11}, \ldots, b_{nn})) \leq \rho(A) \max_{i=1,\ldots,n} b_{ii}, \tag{7}$$

which yields (5) immediately. To obtain (4), set $A = J_n$ where J_n is the $n \times n$ matrix of all ones, in the first inequality of (7). Then

$$\rho(B) \leq \rho(J_n \mathrm{diag}\,(b_{11}, \ldots, b_{nn})) = \mathrm{tr}\, B,$$

since the nonnegative matrix $J_n \mathrm{diag}\,(b_{11}, \ldots, b_{nn})$ is at most rank 1.

If B is diagonally dominant of its row entries, consider $A^T \circ B^T = (A \circ B)^T$ and use the fact that the spectrum is invariant under transpose.

The proof of the second conclusion is similar.

REMARKS:

- The upper bound $\mathrm{tr}\, B$ in (4) is attainable by choosing $B = J_n$.
- Though $\max_{i=1,\ldots,n} b_{ii} \leq \rho(B)$ is true for all nonnegative B, $\rho(B) \leq \mathrm{tr}\, B$ in (4) may not hold if the assumption in the theorem is dropped, for example, the irreducible

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

- It is not true that if $A \geq 0$ and $B \geq 0$ are both diagonally dominant of its (column) row entries, then $\rho(A \circ B) \leq \max_{i=1,\ldots,n} a_{ii} \max_{i=1,\ldots,n} b_{ii}$, for example,

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1.5 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad A \circ B = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix},$$

  with

$$\rho(A \circ B) \approx 4.6180 > 4 = \max_{i=1,2} a_{ii} \max_{i=1,2} b_{ii}.$$

The following provides a representation of the maximum diagonal entry of B.

COROLLARY 1. Let $B \geq 0$ be an $n \times n$ nonnegative matrix. If there exists a positive diagonal D such that DBD^{-1} is diagonally dominant of its column (or row) entries, then

$$\max\{\rho(A \circ B) : A \geq 0,\ \rho(A) = 1\} = \max_{i=1,\ldots,n} b_{ii}.$$

PROOF. Use Theorem 2 (1) and consider $A = I_n$, the $n \times n$ identity matrix.

Let $Z_n := \{A \in \mathbb{R}_{n\times n} : a_{ij} \le 0, i \ne j\}$. A matrix $A \in Z_n$ is called an *M-matrix* [1, 6] if there exists an $P \ge 0$ and $s > 0$ such that

$$A = sI_n - P \quad \text{and} \quad s > \rho(P),$$

where $\rho(P)$ is the spectral radius of the nonnegative P, I_n is the $n \times n$ identity matrix. Denote by $\mathcal{M}_n$ the set of all $n \times n$ nonsingular M-matrices. The matrices in $\mathcal{M}_n^{-1} := \{A^{-1} : A \in \mathcal{M}_n\}$ are called inverse M-matrices. It is known that $A \in Z_n$ is in $\mathcal{M}_n^{-1}$ if and only if A is nonsingular and $A \ge 0$ [6, p.114-115]. It is clear that $\mathcal{M}_n$ and $\mathcal{M}_n^{-1}$ are invariant under similarity via a positive diagonal matrix D, i.e., $D\mathcal{M}_nD^{-1} = \mathcal{M}_n$ and $D\mathcal{M}_n^{-1}D^{-1} = \mathcal{M}_n^{-1}$. It is also known that $\mathcal{M}_n^{-1}$ is Hadamard-closed if and only if $n \le 3$ [8].

Compare the following with [6, p.377].

COROLLARY 2. Let $A \ge 0$, $B \ge 0$ be $n \times n$ nonnegative matrices. If $B \in \mathcal{M}_n^{-1}$, then

$$\rho(B) \le \operatorname{tr} B,$$

and

$$\rho(A \circ B) \le \rho(A) \max_{i=1,\dots,n} b_{ii}.$$

Hence if $B \in \mathcal{M}_n^{-1}$, then

$$\max\{\rho(A \circ B) : A \ge 0,\ \rho(A) = 1\} = \max_{i=1,\dots,n} b_{ii}.$$

PROOF. Since $B^{-1} \in \mathcal{M}_n$, there exists a positive diagonal D such that $DB^{-1}D^{-1}$ is strictly row diagonally dominant [6, p.114-115]. Then the inverse DBD^{-1} is strictly diagonally dominant of its column entries [6, p.125], [9, Lemma 2.2] (or see Proposition 1 in the next section). Then apply Theorem 2 (1).

3 A Sharper Upper Bound When B^{-1} is an M-Matrix

The inequality in Corollary 2 has a resemblance of [6, Theorem 5.7.31, p.375] which asserts that if $A, B \in \mathcal{M}_n$, then

$$\tau(A \circ B^{-1}) \ge \tau(A) \min_{i=1,\dots,n} \beta_{ii}, \tag{8}$$

where

$$\tau(A) = \min\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\},$$

and $\sigma(A)$ denotes the spectrum of $A \in \mathcal{M}_n$. It is known that [6, p.129-130]

$$\tau(A) = \frac{1}{\rho(A^{-1})},$$

and is a positive real eigenvalue of $A \in \mathcal{M}_n$. The number $\tau(A)$ is often called the *minimum eigenvalue* of the M-matrix A. Indeed

$$\tau(A) = s - \rho(P),$$

if $A = sI_n - P$ where $s > \rho(P)$, $P \geq 0$ [6, p.130]. So $\tau(A)$ is a measure of how close $A \in \mathcal{M}_n$ to be singular.

It is known that $A \circ B^{-1} \in \mathcal{M}_n$ if $A, B \in \mathcal{M}_n$ [6, p.359]. Chen [2] provides a sharper lower bound for $\tau(A \circ B^{-1})$ which clearly improves (8):

$$\tau(A \circ B^{-1}) \geq \tau(A)\tau(B) \min_{i=1,\ldots,n} \left[\left(\frac{a_{ii}}{\tau(A)} + \frac{b_{ii}}{\tau(B)} - 1 \right) \frac{\beta_{ii}}{b_{ii}} \right]. \qquad (9)$$

Since $a_{ii} > \tau(A)$ for all $i = 1, \ldots, n$ [1, p.159], (9) implies (8) immediately. Inequality (9) may be rewritten in the following form:

$$\rho((A \circ B^{-1})^{-1}) \leq \frac{\rho(A^{-1})\rho(B^{-1})}{\min_{i=1,\ldots,n}[(a_{ii}\rho(A^{-1}) + b_{ii}\rho(B^{-1}) - 1)\frac{\beta_{ii}}{b_{ii}}]}.$$

However it is not an upper bound for $\rho(A \circ B^{-1})$. In view of Corollary 2 and motivated by Chen's bound and its proof, we provide a sharper upper bound for $\rho(A \circ B)$, where $A \geq 0$ and $B \in \mathcal{M}_n^{-1}$.

We will need a lemma in [9, Lemma 2.2] (a weaker version is found in [7, Lemma 2.2]). The following is a slight extension since if A is a strictly diagonally dominant matrix, then it is nonsingular by the well-known Gersgorin theorem [6, p.31].

PROPOSITION 1. Let $A \in \mathbb{C}_{n \times n}$ be a diagonally dominant nonsingular matrix by row (column respectively), i.e.,

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \qquad \left(|a_{ii}| \geq \sum_{j \neq i} |a_{ji}|, \text{ respectively}\right),$$

for all $i = 1, \ldots, n$. If $A^{-1} = (b_{ij})$, then

$$|b_{ji}| \leq \frac{\sum_{k \neq j} |a_{jk}|}{|a_{jj}|} |b_{ii}| \qquad \left(|b_{ij}| \leq \frac{\sum_{k \neq j} |a_{kj}|}{|a_{jj}|} |b_{ii}|, \text{ respectively}\right).$$

PROOF. Let $A \in \mathbb{C}_{n \times n}$ be a diagonally dominant nonsingular matrix by row. Each $a_{ii} \neq 0$ otherwise the whole i-th row would be zero and then A would be singular. Apply [6, p.129, Problem 17(a)] to yield

$$|b_{ji}| \leq \frac{\sum_{k \neq j} |a_{jk}|}{|a_{jj}|} |b_{ii}|.$$

Consider A^T for the column version.

THEOREM 2. Suppose $A \geq 0$, $B \in \mathcal{M}_n^{-1}$. (i) If A is nilpotent, i.e., $\rho(A) = 0$, then $\rho(A \circ B) = 0$. (ii) If A is not nilpotent, then

$$\rho(A \circ B) \leq \frac{\rho(A)}{\rho(B)} \max_{i=1,\ldots,n} \left[\left(\frac{a_{ii}}{\rho(A)} + \beta_{ii}\rho(B) - 1 \right) \frac{b_{ii}}{\beta_{ii}} \right] \leq \rho(A) \max_{i=1,\ldots,n} b_{ii}.$$

PROOF. (i) If A is nilpotent nonnegative, then A is permutationally similar to a strictly upper triangular (nonnegative) matrix by [4, Theorem 3]. So is $A \circ B$. Hence $\rho(A \circ B) = 0$. (ii) For the second inequality, observe $\max_{i=1,\ldots,n} a_{ii} \leq \rho(A)$ for the nonnegative matrix A, $b_{ii} \geq 0$ and [1, p.159]

$$\beta_{ii} > \tau(B^{-1}) > 0 \qquad \text{for all} \quad i = 1, \ldots, n, \tag{10}$$

since $B^{-1} \in \mathcal{M}_n$. For the first inequality we essentially follow the idea of Chen's proof. Since $\rho(\cdot)$, Hadamard product, and taking inverse are continuous functions, we may assume that $A \geq 0$ and $B^{-1} = \alpha I - P \in \mathcal{M}_n$, where $P \geq 0$ with $\alpha > \rho(P)$, are irreducible (thus $B \geq 0$ is also irreducible since irreducibility is invariant under inverse operation), otherwise we place sufficiently small values $\epsilon > 0$ in the positions in which A and P have zeroes.

Let v and w be the right Perron eigenvectors of P^T and A respectively, i.e., $v, w \in \mathbb{R}^n$ are positive vectors such that

$$(B^{-1})^T v = \tau(B^{-1})v, \qquad Aw = \rho(A)w.$$

So $v^T B^{-1} = \tau(B^{-1})v^T$, or equivalently,

$$\sum_{k=1}^{n} v_k \beta_{kj} = \tau(B^{-1})v_j, \quad j = 1, \ldots, n.$$

Define

$$C := VB^{-1}, \qquad V := \operatorname{diag}(v_1, \ldots, v_n).$$

Since the off diagonal entries β_{ij}, $i \neq j$, of B^{-1} are nonpositive and $\tau(B^{-1}) > 0$, C is strictly diagonally dominant by column. Apply Proposition 1 on $C^{-1} = BD^{-1}$ to have

$$\frac{b_{ij}}{v_j} = \frac{|b_{ij}|}{|v_j|} \leq \frac{\sum_{k \neq j} |v_k \beta_{kj}|}{|v_j \beta_{jj}|} \cdot \frac{|b_{ii}|}{|v_i|} = \frac{-\sum_{k \neq j} v_k \beta_{kj}}{v_j \beta_{jj}} \cdot \frac{b_{ii}}{v_i} = \frac{(\beta_{jj} - \tau(B^{-1}))v_j}{v_j \beta_{jj}} \cdot \frac{b_{ii}}{v_i},$$

for all $i \neq j$, since $v_j > 0$, $\beta_{kj} \leq 0$ $(k \neq j)$, $\beta_{jj} > 0$. Hence for $i \neq j$,

$$b_{ij} \leq \frac{(\beta_{jj} - \tau(B^{-1}))v_j b_{ii}}{\beta_{jj} v_i}.$$

Now define a positive vector $z \in \mathbb{R}^n$:

$$z_i := \frac{w_i \beta_{ii}}{v_i(\beta_{ii} - \tau(B^{-1}))} > 0, \quad i = 1, \ldots, n.$$

Then

$$\begin{aligned}[(A\circ B)z]_i &= a_{ii}b_{ii}z_i + \sum_{j\neq i} a_{ij}b_{ij}z_j \\ &\leq a_{ii}b_{ii}z_i + \sum_{j\neq i} a_{ij}\frac{(\beta_{jj}-\tau(B^{-1}))v_j b_{ii}}{\beta_{jj}v_i}\cdot\frac{w_j\beta_{jj}}{v_j(\beta_{jj}-\tau(B^{-1}))} \\ &= a_{ii}b_{ii}z_i + \frac{b_{ii}}{v_i}\sum_{j\neq i} a_{ij}w_j \\ &= a_{ii}b_{ii}z_i + \frac{b_{ii}}{v_i}(\rho(A)-a_{ii})w_i \\ &= b_{ii}z_i\left[a_{ii}+\frac{1}{\beta_{ii}}(\rho(A)-a_{ii})(\beta_{ii}-\tau(B^{-1})\right] \\ &= \frac{b_{ii}}{\beta_{ii}}\rho(A)\tau(B^{-1})\left[\frac{a_{ii}}{\rho(A)}+\frac{\beta_{ii}}{\tau(B^{-1})}-1\right]z_i \\ &\leq \rho(A)\tau(B^{-1})\max_{i=1,\dots,n}\left[\left(\frac{a_{ii}}{\rho(A)}+\frac{\beta_{ii}}{\tau(B^{-1})}-1\right)\frac{b_{ii}}{\beta_{ii}}\right]z_i.\end{aligned}$$

By [1, p.28, Theorem 2.1.11], we have the desired result, since $\tau(B^{-1}) = 1/\rho(B)$. Now by (2) we have

$$\left(\frac{a_{ii}}{\rho(A)}+\beta_{ii}\rho(B)-1\right)\frac{b_{ii}}{\beta_{ii}} \leq \beta_{ii}\rho(B)\frac{b_{ii}}{\beta_{ii}} = \rho(B)b_{ii}, \quad i=1,\dots,n.$$

So the second inequality (see Corollary 2) follows immediately.

We reamrk that the first inequality in Theorem 2 is no long true if we merely assume that B is nonsingular nonnegative. For example, if

$$A=\begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}, B=\begin{pmatrix}1&2&0\\0&1&2\\2&0&1\end{pmatrix}, A\circ B=\begin{pmatrix}0&2&0\\0&0&2\\2&0&0\end{pmatrix}, B^{-1}=\frac{1}{9}\begin{pmatrix}1&-2&4\\4&1&-2\\-2&4&1\end{pmatrix},$$

then $\rho(A)=1$, $\rho(B)=3$, $\rho(A\circ B)=2$, but

$$\frac{\rho(A)}{\rho(B)}\max_{i=1,\dots,n}\left[\left(\frac{a_{ii}}{\rho(A)}+\beta_{ii}\rho(B)-1\right)\frac{b_{ii}}{\beta_{ii}}\right]=-2.$$

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References

[1] A. Berman and R.J. Plemmons, Nonnegative Matrices in the Mathematical Sciences, Classics in Applied Mathematics, 9, SIAM, Philadelphia, 1994.

[2] S. C. Chen, A lower bound for the minimum eigenvalue of the Hadamard product of matrices, Linear Algebra Appl., 378(2004), 159–166.

[3] L. Elsner, C. R. Johnson and J. A. Dias da Silva, The Perron root of a weighted geometric mean of nonnegative matrices, Linear and Multilinear Algebra, 24(1988), 1–13.

[4] J. Z. Hearon, Compartmental matrices with single root and nonnegative nilpotent matrices, Math. Biosci., 14(1972), 135–142.

[5] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University Press, 1985.

[6] R. A. Horn and C. R. Johnson, Topics in Matrix Analysis, Cambridge University Press, 1991.

[7] Y. Song, On an inequality for the Hadamard product of an M-matrix and its inverse, Linear Algebra Appl., 305(2000), 99–105.

[8] B. Y. Wang, X. Zhang and F. Zhang, On the Hadamard product of inverse M-matrices, Linear Algebra Appl., 305(2000), 23–31.

[9] X. Yong and Z. Wang, On a conjecture of Fiedler and Markham, Linear Algebra Appl., 288(1999), 259–267.