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MINIMAL OPERATION TIME OF ENERGY DEVICES*

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Abstract

We consider the problem of determining the minimal time for which an energy supply source should operate in order to supply a system with a desired amount of energy in finite time.

1 Introduction

While boiling water or any other liquid, most of us have noticed that the heater can be switched off at an intuitively chosen time and the liquid will still reach its boiling point no long after the heater is switched off. A natural question arises: how can switch-off time be chosen in an optimal way so that electrical energy will not be wasted? In other words what is the earliest time at which the heater can be turned off while still reaching the liquid's boiling point in a finite time? A general formulation of the problem is as follows: Let D be a device that supplies energy to a system S through a supply line. Let $E'(t)$ be the energy supply rate. We assume that D can be switched on and off and that it continues to supply energy, at a decreasing rate, for some time after it has been switched off. Question: What is the minimum switch-off time of D (corresponding to the minimal operational time of D) in order to transfer to S a total amount of energy Q (where $Q > 0$ is given)? The device D can also be viewed as a control mechanism for bringing the system S from an energy level E_1 to a higher energy level E_2 in finite time while operating for the minimum time possible. Clearly, the solution of this problem can have a lot of applications, both civilian (e.g. energy conservation) and military (e.g. minimizing detection risk).

2 Examples

EXAMPLE 1 (Exponential Model). We consider a simple example of an energy device supplying energy to a system at time $t \geq 0$ at a rate (which in what follows we consider

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to include the rate at which the transferred energy is possibly radiating from the system and/or the supply line) given by

$$E'(t) = \begin{cases} e^{a\,t} - 1 & \text{if } 0 \leq t \leq t_0 \\ e^{a\,t_0} - 1 & \text{if } t_0 \leq t \leq t_1 \\ \frac{e^{a\,t_0}-1}{1-e^{-b\,T}} \left(e^{-b\,(t-t_1)} - e^{-b\,T}\right) & \text{if } t \geq t_1 \end{cases}$$

where a and b are positive real numbers characteristic of the source but also depending on the environment, t_0 is the time at which the energy supply rate is at its peak, and t_1 is the switch-off time. We assume that the optimal switch-off time is after the rate of energy supply has been stabilized i.e. that $\hat{t_1} \geq t_0$. We also assume that after switching-off at time $t_1 \geq t_0$, the source stops transferring energy to the device at time $t_1 + T$, where $T > 0$ is independent of t_1. Let $Q > 0$ be the amount of energy that we wish to transfer to the system. We assume that the transfer of this amount of energy will occur at some time $t_2 \geq t_1 \geq t_0$. The energy Q could be, for example, the energy required to bring a liquid substance to its boiling temperature, or the energy required for complete phase transition. In the latter case $Q = m\,L_v$, where m is the mass of the liquid and L_v is its latent heat of vaporization (cf. [2]). We require that

$$\int_0^{t_2} E'(s)ds = Q$$

which implies that

$$\int_0^{t_0} E'(s)\,ds + \int_{t_0}^{t_1} E'(s)ds + \int_{t_1}^{t_2} E'(s)\,ds = Q$$

or, by the definition of $E'(s)$,

$$\int_0^{t_0} (e^{a\,s} - 1)\,ds + \int_{t_0}^{t_1} \left(e^{a\,t_0} - 1\right)\,ds + \int_{t_1}^{t_2} \frac{e^{a\,t_0} - 1}{1 - e^{-b\,T}} \left(e^{-b\,(s-t_1)} - e^{-b\,T}\right) ds = Q$$

which implies that

$$\left[\frac{e^{a\,s}}{a} - s\right]_{s=0}^{s=t_0} + \left(e^{a\,t_0} - 1\right)\,(t_1 - t_0) + \frac{e^{a\,t_0} - 1}{1 - e^{-b\,T}} \left[-\frac{e^{-b\,(s-t_1)}}{b} - e^{-b\,T}\,s\right]_{s=t_1}^{s=t_2} = Q$$

from which letting

$$\begin{aligned} y &: = t_2 - t_1 \\ L_0 &: = \frac{1}{e^{a\,t_0} - 1} \left(Q + \frac{1}{a} - \frac{e^{a\,t_0}}{a} + e^{a\,t_0}\,t_0 + \frac{1 - e^{a\,t_0}}{1 - e^{-b\,T}}\,\frac{1}{b}\right) \\ L_1 &: = \frac{1}{b(1 - e^{-b\,T})} \\ L_2 &: = \frac{1}{1 - e^{-b\,T}} \end{aligned}$$

we obtain

$$t_1 = L_0 + L_1\, e^{-b\,y} + L_2 y. \tag{1}$$

Notice that if $y = 0$ then $t_1 = L_0 + L_1$ is the time the device supplies the desired energy level Q without being switched-off. The maximum value of y is $y = y_{max} = (t_1 + T) - t_1 = T$. The optimal switch-off time $\hat{t_1}$ is therefore determined from (1) by letting $y = T$ and it is given by

$$\hat{t_1} = L_0 + L_1 e^{-b\,T} + L_2\, T.$$

Energy level Q is reached at time

$$t_2 = \hat{t_1} + T = L_0 + L_1\, e^{-b\,T} + (L_2 + 1)T.$$

EXAMPLE 2 (Linear Model). A simplified version of the previous model is obtained by assuming that

$$E'(t) = \begin{cases} \frac{a}{t_0}\, t & \text{if } 0 \le t \le t_0 \\ a & \text{if } t_0 \le t \le t_1 \\ -\frac{a}{T}(t - t_1) + a & \text{if } t \ge t_1 \end{cases}$$

where $t_0 > 0$, $a > 0$, and T is as in the model above. In this case

$$\int_0^{t_2} E'(s) ds = Q$$

implies that

$$\int_0^{t_0} E'(s)\, ds + \int_{t_0}^{t_1} E'(s) ds + \int_{t_1}^{t_2} E'(s)\, ds = Q$$

or, by the definition of $E'(s)$,

$$\int_0^{t_0} \frac{a}{t_0} s ds + \int_{t_0}^{t_1} a ds + \int_{t_1}^{t_2} \left(-\frac{a}{T}\,(t - t_1) + a\right)\, ds = Q$$

which implies that

$$\left[\frac{a}{t_0}\,\frac{s^2}{2}\right]_{s=0}^{s=t_0} + a\,(t_1 - t_0) + \left[-\frac{a}{2\,T}\,(s - t_1)^2 + a\,s\right]_{s=t_1}^{s=t_2} = Q$$

and letting $y = t_2 - t_1$ we obtain

$$t_1 = \frac{1}{2T} y^2 - y + \frac{Q}{a} + \frac{t_0}{2}.$$

As in Example 1, substituting y by $y_{max} = T$ we obtain

$$\hat{t_1} = -\frac{1}{2} T + \frac{Q}{a} + \frac{t_0}{2}$$

which is greater than or equal to t_0 if and only if $\frac{Q}{a} - \frac{1}{2}T - \frac{t_0}{2} \ge 0$. Energy level Q is reached at time

$$t_2 = \hat{t_1} + T = \frac{t_0}{2} + \frac{Q}{a} + \frac{1}{2}\, T.$$

3 General Description of the Optimal Switch-Off Time

The examples treated in detail in the previous section suggest the following general theorems.

THEOREM 1. Let, for each $t_1 \geq t_0$,

$$E'_{t_1}(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq t_0 \\ f(t_0) & \text{if } t_0 \leq t \leq t_1 \\ g(t) & \text{if } t \geq t_1 \end{cases}$$

where f is continuous and increasing with $f(0) = 0$, and g is continuous and decreasing with $g(t_1) = f(t_0)$. Let F and G denote the anti-derivatives of f and g respectively. If there exists $T > 0$ such that $g(t_1 + T) = 0$ for all $t_1 \geq t_0$, then the optimal switch-off time $\hat{t_1} \geq t_0$ is the solution of

$$F(t_0) + f(t_0)\,(\hat{t_1} - t_0) + G(\hat{t_1} + T) - G(\hat{t_1}) = Q.$$

PROOF. The condition

$$\int_0^{t_2} E'_{t_1}(s)\,ds = Q$$

implies that

$$(F(t_0) - F(0)) + f(t_0)(t_1 - t_0) + (G(t_2) - G(t_1)) = Q$$

which by $F(0) = 0$ and the fact that for the optimal $\hat{t_1}$, $t_2 = \hat{t_1} + T$ implies that

$$F(t_0) + f(t_0)\,(\hat{t_1} - t_0) + G(\hat{t_1} + T) - G(\hat{t_1}) = Q.$$

We can generalize the above theorem to the case of a heat source that supplies energy at a possibly strictly increasing (or even arbitrary continuous) rate as follows.

THEOREM 2. Let, for each $t_1 \geq t_0$, the energy function $E_{t_1}(t)$ be an increasing continuously differentiable function of t. The optimal switch-off time $\hat{t_1}$ and the associated time t_2 at which the energy level reaches Q are determined by the system of equations

$$E_{\hat{t_1}}(t_2) = Q, \quad E'_{\hat{t_1}}(t_2) = 0.$$

PROOF. It is clear that the optimal switch-off time $\hat{t_1}$ makes full use of the energy source in the sense that the desired energy amount Q is supplied at the moment when the source dies out i.e.e when $E'_{\hat{t_1}}(t_2) = 0$. For such a t_2,

$$\int_0^{t_2} E'_{\hat{t_1}}(s)\,ds = Q$$

implies

$$\int_0^{\hat{t_1}} E'_{\hat{t_1}}(s)ds + \int_{\hat{t_1}}^{t_2} E'_{\hat{t_1}}(s)ds = Q$$

i.e.e $E_{\hat{t_1}}(\hat{t_1}) - E_{\hat{t_1}}(0) + E_{\hat{t_1}}(t_2) - E_{\hat{t_1}}(\hat{t_1}) = Q$ and since $E_{\hat{t_1}}(0) = 0$, we obtain $E_{\hat{t_1}}(t_2) = Q$.

4 Noisy Supply Line

The energy supply rate functions $\phi_{t_1}(t) := E'(t)$ used in Examples 1 and 2 above, can be viewed on each of the intervals $[0, t_0]$, $[t_0, t_1]$ and $[t_1, t_2]$, as solutions of ordinary differential equations of the form

$$d\phi_{t_1}(t) = (c_1\,\phi_{t_1}(t) + c_2)dt \tag{2}$$

for appropriate constants c_1, c_2 (depending on the interval) and initial condition $\phi_{t_1}(0) = 0$, with the different branches of $\phi_{t_1}(t)$ tied up at t_0 and t_1. It could be the case however, that the supply line connecting the energy source D with the system S is affected by noise, appearing in (2) in the form of additive noise

$$d\phi_{t_1}(t) = (c_1\,\phi_{t_1}(t) + c_2)dt + (c_3\phi_{t_1}(t) + c_4)\,dB(t) \tag{3}$$

or equivalently

$$\phi_{t_1}(t) = \int_0^t (c_1\,\phi_{t_1}(s) + c_2)\,ds + \int_0^t (c_3\phi_{t_1}(s) + c_4)\,dB(s) \tag{4}$$

where c_1, c_2, c_3, c_4 are constants, $B(s)$ is one dimensional Brownian motion and the stochastic integral on the right hand side of (4) is in the sense of Itô (cf. [1]). In that case $\phi_{t_1}(t)$ is actually a stochastic process $\phi_{t_1}(t, \omega)$ and the problem of finding the optimal switch-off time $\hat{t_1}$ now amounts to finding the first t_1 for which

$$\int_0^{t_2} \mu_{t_1}(s)ds = Q$$

for some finite $t_2 \geq t_1$, where $\mu_{t_1}(t)$ denotes the mathematical expectation of $\phi_{t_1}(t, \omega)$. Since the mathematical expectation of an Itô stochastic integral with respect to Brownian motion is equal to zero, (4) implies upon taking the expectation of both sides that

$$\mu_{t_1}(t) = \int_0^t (c_1\,\mu_{t_1}(s) + c_2)\;ds$$

which can be solved explicitly and yields the mean energy supply rate

$$\mu_{t_1}(t) = \frac{c_2}{c_1}\,\left(e^{c_1\,t} - 1\right)$$

where on each interval the constants c_1 and c_2 are determined by using the initial and tying up conditions. Thus we are reduced to the deterministic problem considered in the examples of Section 2, but this time for the mean energy supply function. The method extends directly to the case when equations (2) and (3) are replaced by the more general equations

$$d\phi_{t_1}(t) = f(t, \phi_{t_1}(t))\,dt$$

and

$$d\phi_{t_1}(t) = f(t, \phi_{t_1}(t))\,dt + g(t, \phi_{t_1}(t))dB(t)$$

respectively.

References

[1] B. Oksendal, Stochastic Differential Equations, 2nd edition, Springer-Verlag 1989.

[2] J. S. Faughn, R. A. Serway, C. Vuille and C. A. Bennett, College Physics, 7th edition, Thomson Brooks/Cole 2006.