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A PRIORI ESTIMATES FOR SOLUTIONS OF SPINODAL DECOMPOSITION PROBLEM*

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Abstract

We show the existence of smooth solutions of a nonlinear partial differential equation modeling the dynamics of spinodal decomposition.

1 Introduction

In this paper, we consider an equation modeling phase separation in spinodal decomposition dynamics, which takes place in solid and liquid solutions under specific thermodynamical conditions. The initial stages of phase separation is revealed in the traditional Cahn-Hilliard theory [8]. However, Aifantis and Serrin in [3] suggested a generalization of the Cahn-Hilliard theory by including additional terms of interfacial stress. The derivation of their model equations is based on the balance laws of mass and momentum

$$\begin{cases} u_t + \nabla \cdot J = 0 \\ \nabla \cdot T = F \end{cases} \tag{1}$$

where u is the concentration, J is the flux of the diffusing material, F is the diffusive force and T is the symmetric stress tensor which includes the interfacial terms of a typical liquid-vapor phase transition. Combining the components of the tensor T in the one-dimensional case leads to the expression

$$T = -p(u) + \varepsilon u_x^2 + \delta u_{xx} \tag{2}$$

where δ is an interfacial coefficient and ε is a short range deformity coefficient. Generally, these coefficients are functions of concentration, but in this work, we assume them to be constants. The equation of state $p(u)$ is assumed to be nonconvex with a cubic like form. The diffusive force F can be taken to be proportional to the flux J and its time rate, to incorporate inertia effects, that is

$$F = M^{-1}J + mJ_t \tag{3}$$

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where M is a mobility coefficient and m is a constant measuring the effect of inertia, which will be dropped in the current model, to obtain the relation

$$J = -M(p(u) - \varepsilon u_x^2 - \delta u_{xx})_x \tag{4}$$

Furthermore, we assume that the time dependent tensor T contains viscous relaxation terms of the form νu_t. Combining the relaxation terms and letting $M = 1$ in (4) from the one-dimensional mass balance equation (1) we arrive at the equation

$$u_t = p(u)_{xx} + \nu u_{xxt} - \varepsilon(u_x^2)_{xx} - \delta u_{xxxx} \tag{5}$$

This is a fourth order nonlinear differential equation, solutions of which may explain the later stages of the transient spinodal decomposition process. Many aspects of the Cahn-Hilliard equation have been studied by Bates and Fife [4], Temam [11] and Witelski [12]. The stationary and mechanical solutions of (5) have been investigated by Aifantis and Serrin in [2, 3]. In the current paper, we discuss the solvability of the equation (5) for various boundary conditions, which are needed to explain the long time behavior and the evolution of spinodal decomposition process.

2 A Priori Estimates

We consider the nonlinear evolution equation of spinodal decomposition for the density $u = u(x,t)$, that is

$$u_t = p(u)_{xx} + \nu u_{xxt} - \varepsilon(u_x^2)_{xx} - \delta u_{xxxx} \tag{6}$$

on a bounded domain $\Omega = [0, l]$ with the initial condition

$$u(0,x) = u_0(x) \in H^2(\Omega) \tag{7}$$

The equation (6) is supplemented with either periodic boundary conditions (see Temam [9]):

$$\frac{\partial^i u}{\partial x^i}(0,t) = \frac{\partial^i u}{\partial x^i}(l,t) \quad \text{for } i = 0,1,2,3 \tag{8}$$

or Neumann boundary conditions, see Bates and Fife [4]:

$$\frac{\partial u}{\partial x}(x,t) = \frac{\partial^3 u}{\partial x^3}(x,t) = 0 \quad \text{for } x = 0, l. \tag{9}$$

In addition, we will assume that the equation of state $p(u) \in C^{(3)}$ and grows linearly for $|u| > N$ for some large positive number N and $p'(u)$ changes its sign inside an interval of displacement (phase separation).

2.1 Local Existence

The derivation of local existence can be found for general differential equations in Henry [9], also presented briefly in Zheng [10]. Along the outline proof of local existence in time (see Henry [9]), we partition the differential operators into auxiliary linearized part and the remaining terms as follows:

$$u_t = [-(1-\nu\partial_x^2)^{-1}\partial_x^2](\delta u_{xx}) + [(1-\nu\partial_x^2)^{-1}\partial_x^2](p(u) + \varepsilon(u_x^2)) \tag{10}$$

We apply Fourier transform to the equation and treat the linear part as heat operator and denote the nonlinear part by f to obtain

$$\hat{u} = e^{-\delta\frac{\xi^4}{1+\nu\xi^2}t}\hat{f}(\xi) \tag{11}$$

We apply the contraction mapping theorem for sufficiently small time and energy estimation for the nonlinear part.
However the crucial step in proving the global existence is to have a priori uniform estimates of the solution for any time $T < +\infty$ followed by the continuation argument (see Zheng [10]).

2.2 Global Existence

Throughout this paper, $\|.\|$ will denote the $L^2(\Omega)$ norm and $c > 0$ will denote a generic constant that might depend on the initial data, ε, δ, ν, and possibly T but independent of t. The arguments in integrals will be omitted if they are clear. We prove the following theorem:

MAIN THEOREM. The equation (6) with the initial conditions (7) and boundary conditions (8) or (9), has a global solution $u \in C([0,T]; H(\Omega))$.

PROOF. The proof of this theorem is based on establishing a priori uniform estimates on the solution u. We group these estimates into two lemmas.

LEMMA 2.1. For any $t \in [0,T]$ we have

$$\sup_{0\le t\le T}(\|u\|^2 + \|u_x\|^2) + \int_0^T \|u_{xx}\|^2 dt \le c. \tag{12}$$

PROOF. Multiply equation (6) by u and integrate by parts over Ω to obtain:

$$\int_0^l uu_t dx = \int_0^l up_{xx} dx + \nu\int_0^l uu_{txx} dx - \varepsilon\int_0^l u(u_x^2)_{xx} dx - \delta\int_0^l uu_{xxxx} dx \tag{13}$$

We evaluate each term using the boundary conditions (8) or (9) and the restrictions on the pressure p, as follows

$$\begin{aligned} \int_0^l uu_{txx}dx &= uu_{tx}\Big|_0^l - \int_0^l u_x u_{tx}dx = -\frac{1}{2}(\|u_x\|^2)_t. \\ -\int_0^l uu_{xxxx}dx &= -uu_{xxx}\Big|_0^l + \int_0^l u_x u_{xxx}dx = \\ &= u_x u_{xx}\Big|_0^l - \int_0^l u_{xx}^2 dx = -\|u_{xx}\|^2. \end{aligned} \tag{14}$$

$$\begin{aligned} -\int_0^l u(u_x^2)_{xx}dx &= -u(u_x^2)_x\Big|_0^l + \int_0^l u_x u(^2_x)_x dx = \\ &= -\int^l u_x^2 u_{xx}dx = -\frac{1}{3}u_x^3\Big|_0^l = 0. \\ \left|\int_0^l up_{xx}dx\right| &= \left|(up_x)|_0^l - \int_0^l p'(u)u_x^2 dx\right| \le k\int_0^l u_x^2 dx = k\|u_x\|^2. \end{aligned}$$

We combine these estimates to obtain

$$\frac{1}{2}(\|u\|^2 + \nu\|u_x\|^2)_t + \delta\|u_{xx}\|^2 \le k\|u_x\|^2. \tag{15}$$

Integrate the inequality (15) over time interval $[0,T]$ and invoke Gronwall's lemma to deduce the required estimate (12).

LEMMA 2.2. There holds

$$\sup_{0\le t\le T}(\|u_t\|^2 + \|u_{tx}\|^2) + \int_0^T \|u_{txx}\|dt \le c. \tag{16}$$

PROOF. Differentiate (6) with respect to t and multiply by u_t to get

$$u_t u_{tt} = u_t p_{xxt} + \nu u_t u_{xxtt} - \varepsilon u_t(u_x^2)_{xxt} - \delta u_t u_{xxxxt}. \tag{17}$$

Integrating (17) by parts over Ω taking into account the boundary conditions and the estimates from lemma 2.1 yields the following relations

$$\begin{aligned} \int_0^l u_t u_{xxtt}dx &= (u_t u_{xtt})|_0^l - \int_0^l u_{xt}u_{xtt}dx = \\ &= -\frac{1}{2}\int_0^l u_{xt}^2 dx = -\frac{1}{2}\|u_{xt}\|^2. \\ -\int_0^l u_t(u_x^2)_{xxt}dx &= (u_t(u_x^2)_{xt})|_0^l + \int^l u_{xt}(u_x^2)_{xt}dx = \\ &= -\int_0^l u_{xxt}(u_x^2)_t dx = \\ &= -2\int_0^l u_{xxt}u_x u_{xt}dx. \end{aligned} \tag{18}$$

Apply the Cauchy-Schwartz inequality to the last integral to get

$$\begin{aligned} \left|\int_0^l u_t(u_x^2)_{xxt}dx\right| &\leq 2\chi\int_0^l u_{xxt}^2 + \frac{4}{\chi}\int_0^l u_x^2u_{xt}^2dx \\ &\leq 2\chi\|u_{xxt}\|^2 + \frac{8}{\chi}\|u_{xt}\|_{L^\infty}^2\int_0^l u^2dx \qquad (19) \\ &\leq 2\chi\|u_{xxt}\|^2 + \frac{8c}{\chi}\|u_{xt}\|_{L^\infty}^2\int_0^l u_x^2dx. \end{aligned}$$

We evaluate the last term in (19) by applying the Young inequality

$$\begin{aligned} \|u_{xt}\|_{L^\infty} &\leq c_1\|u_{xxt}\|^{1/2}\|u_{xt}\|^{1/2} + c_2\|u_{xt}\| \\ &\leq c_1\chi^2\|u_{xxt}\| + \frac{c_1}{4\chi^2}\|u_{xt}\| + c_2\|u_{xt}\|. \qquad (20) \end{aligned}$$

Regrouping these estimates to obtain

$$\|u_{xt}\|_{L^\infty}^2 \leq c\chi^2\|u_{xxt}\|^2 + c(\chi)\|u_{xt}\|^2. \qquad (21)$$

where $c(\chi)$ is a function of χ. Similarly, we evaluate the remaining terms to obtain

$$\left|\int_0^l u_t p(u)_{xxt}dx\right| \leq \frac{k}{\chi}\|u_t\|^2 + k\chi\|u_{xxt}\|^2 \qquad (22)$$

and

$$\int_0^l u_t u_{xxxxt}dx = \|u_{xxt}\|^2 \qquad (23)$$

Substituting these estimates and selecting a small enough χ we arrive at the following inequality

$$\frac{1}{2}(\|u_t\|^2 + \nu\|u_{xt}\|^2)_t + \delta_0\|u_{xxt}\|^2 \leq c_1\|u_t\|^2 + c_2\|u_{xt}\|^2, \qquad (24)$$

where δ_0 is a positive constant. Integrating (24) over $[0, T]$ and invoking the Gronwall's lemma. We conclude (16).

REMARK. We can derive additional energy estimates for higher order derivatives of the solution of equation (6) by interpolation relations and requiring suitable degree of smoothness of initial data.

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