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ON THE EIGENFUNCTIONS OF A MODIFIED DURRMEYER OPERATOR AND APPLICATION *

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Abstract

We show that the eigenfunctions of a modified Durrmeyer operator are orthogonal polynomials related to Jacobi polynomials. We give as application a fast drawing of a degree n Bézier curve approximation for given $(r+1)$ points, where n does not depend on r and the rate of approximation being $O(n^{-1/2})$.

1 Introduction

Durrmeyer [6] has introduced a Bernstein type operator of degree n defined by

$$M_n(f,x) = (n+1)\sum_{i=0}^{n} b_i^n(x)\int_0^1 f(u)b_i^n(u)du, \tag{1}$$

where $f(u)$ is an integrable function on $[0,1]$ and $b_i^n(t) = \binom{n}{i}t^i(1-t)^{n-i}$, $i = 0,...,n$, are the degree n Bernstein polynomials basis.

Many authors have studied the operator $M_n(f,x)$ [2, 3, 4, 9]. However the operator $M_n(-,x)$ does not possess the property of endpoint interpolation which is essential for interpolation problem. For this reason we consider in this paper a modified kind of the Durrmeyer-Bernstein operator introduced by Goodman and Sharma [7], defined by

$$U_n(f,x) = b_0^n(x)f(0) + (n-1)\sum_{i=1}^{n-1} a_i b_i^n(x) + b_n^n(x)f(1), \quad x \in [0,1], \tag{2}$$

where $a_i = \int_0^1 b_{i-1}^{n-2}(u)f(u)du$.

Similar operators defined by

$$P_n(f,x) = n\sum_{i=1}^{n-1} b_i^n(x)\int_0^1 f(u)b_{i-1}^{n-1}(u)du + (1-x)^n f(0)$$

were recently introduced by Gupta and Maheshwari [10].

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The operator $U_n(f,x)$ satisfies the endpoint conditions

$$U_n(f,0) = f(0), \quad U_n(f,1) = f(1),$$

and have interesting properties [7]. In particular, $U_n(-,x)$ is a linear positive operator such that $U_n(1,x) = 1$, and $\|Un(f) - f\|_\infty = \|f\|_\infty$, where $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$.

In section 2 we establish a common property [2] shared by M_n and U_n by giving the expression of $U_n(t^k,x)$ in terms of the Bernstein basis $b_i^p(x)$, $i = 0,...,p$ with $p = \min(n,k)$. Then we show that the eigenfunctions of the operator U_n are orthogonal polynomials.

In section 3 we present a degree n approximation of a set of $(r+1)$ given points $q_0, q_1, ..., q_r$ in $\mathbf{R}^d$ where n does not depend on r. More precisely we approximate the polygonal line passing through the given points q_i by a Bézier curve using the operator $U_n(f,t)$ expressed in term of the eigenfunctions. This method yields a fast drawing of the Bézier curve $U_n(f,t) = \sum_{i=0}^n p_i b_i^n(t)$ with only $O(n)$ multiplications while the usual de Casteljau algorithm [6] using a repeated linear interpolation of the control points p_i, $i = 0,...,n$ needs $O(n^2)$ multiplications.

2 Some Basic Properties

We give below useful basic properties of the operator U_n. In particular,

- If $f(t)$ is a degree k polynomial with $k \le n$ then $U_n(f,x)$ is a degree k polynomial.
- The eigenfunctions of U_n are orthogonal polynomials.

Goodman and Sharma [7] have given the expression of $U_n(t^k,x)$ in terms of the monomial basis $(1, x, \cdots, x^k)$. We shall show that in fact $U_n(t^k,x)$ is of degree $p = \min(n,k)$ and it can be given in terms of the Bernstein basis $b_i^p(x)$. This property is also shared by M_n and plays an important role in the next section.

PROPOSITION 1. (i). For all integers $n \ge 1$, $k \ge 1$, let $p = \min(n,k)$ and $q = \max(n,k)$. Then

$$\sum_{i=0}^{n} \binom{k+i}{k} b_i^n(t) = \sum_{i=0}^{k} \binom{n+i}{n} b_i^k(t), \tag{3}$$

$$\sum_{i=0}^{n} k\binom{k+i-1}{k} b_i^n(t) = \sum_{i=0}^{k} n\binom{n+i-1}{n} b_i^k(t). \tag{4}$$

(ii). $U_n(t^k,x)$ is a polynomial of degree p such that

$$U_n(t^k,x) = \frac{(n-1)!(k-1)!}{(n+k-1)!} \sum_{i=0}^{p} \alpha_i b_i^p(x), \tag{5}$$

with $\alpha_i = q\binom{q+i-1}{q}$.

PROOF. (i). If $n = k$ the identity (3) is obvious. Assume that $n > k$ and let $u_{n,k}(x,y) = \frac{1}{k!}x^k(x+y)^n$, then we have

$$\begin{aligned}
\frac{\partial^k}{\partial x^k}u_{n,k}(x,y) &= \frac{1}{k!}\frac{\partial^k}{\partial x^k}\sum_{i=0}^{n}\binom{n}{i}x^{k+i}y^{n-i} = \sum_{i=0}^{n}\binom{n}{i}\binom{k+i}{k}x^i y^{n-i} \\
\frac{\partial^n}{\partial x^n}u_{k,n}(x,y) &= \frac{1}{n!}\frac{\partial^n}{\partial x^n}\sum_{i=0}^{k}\binom{k}{i}x^{n+i}y^{k-i} = \sum_{i=0}^{k}\binom{k}{i}\binom{n+i}{k}x^i y^{k-i}.
\end{aligned}$$

For $y = 1 - x$, the above equations give

$$\frac{\partial^k}{\partial x^k}u_{n,k}(x,y) = \sum_{i=0}^{n}\binom{k+i}{k}b_i^n(x), \tag{6}$$

$$\frac{\partial^n}{\partial x^n}u_{k,n}(x,y) = \sum_{i=0}^{k}\binom{n+i}{n}b_i^k(x). \tag{7}$$

On the other hand using Liebnitz formula for derivative we obtain

$$\frac{\partial^k}{\partial x^k}u_{n,k}(x,y) = \frac{1}{k!}\sum_{i=0}^{k}\binom{k}{i}\frac{\partial^{k-i}}{\partial x^{k-i}}x^k\frac{\partial^i}{\partial x^i}(x+y)^n = \sum_{i=0}^{k}\binom{k}{i}\binom{n}{i}x^i(x+y)^{n-i},$$

$$\frac{\partial^n}{\partial x^n}u_{k,n}(x,y) = \frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\frac{\partial^{n-i}}{\partial x^{n-i}}x^n\frac{\partial^i}{\partial x^i}(x+y)^k. \tag{8}$$

The assumption $n > k$ implies that $\frac{\partial^i}{\partial x^i}(x+y)k = 0$ for $i = k+1, ..., n$. Then

$$\frac{\partial^n}{\partial x^n}u_{k,n}(x,y) = \sum_{i=0}^{k}\binom{k}{i}\binom{n}{i}x^i(x+y)^{k-i}. \tag{9}$$

Setting $y = 1 - x$, (8) is equal to (9), therefore (6) is equal to (7) and we get (3). Now using (3) and the degree elevation identity [6]

$$kb_i^{k-1}(x) = (i+1)b_{i+1}^k(x) + (k-i)b_i^k(x),$$

we have

$$\begin{aligned}
\sum_{i=0}^{n}\binom{k+i-1}{k}b_i^n(x) &= \sum_{i=0}^{n}\binom{k+i}{k}b_i^n(x) - \sum_{i=0}^{n}\binom{k+i-1}{k-1}b_i^n(x) \\
&= \sum_{i=0}^{k}\binom{n+i}{n}b_i^k(x) - \sum_{i=0}^{k-1}\binom{n+i}{n}b_i^{k-1}(x) \\
&= \sum_{i=0}^{k}\left[\binom{n+i}{n} - \binom{n+i-1}{n}\frac{i}{k} - \binom{n+i}{n}\frac{k-i}{k}\right]b_i^k(x) \\
&= \frac{n}{k}\sum_{i=0}^{k}\binom{n+i-1}{n}b_i^k(x).
\end{aligned}$$

The proof of part 1 is complete.

For instance we have the following remarkable results:

- for $k = 1$ and $n \geq 1$, $\sum_{i=0}^{n} i b_i^n(x) = nx$,
- for $k = 2$ and $n \geq 2$, $\sum_{i=0}^{n} i(i+1) b_i^n(x) = n b_1^2(x) + n(n+1) b_2(x)$.

(ii). By the definition (2) of the operator U_n, we have

$$U_n(t^k, x) = (n-1) \sum_{i=1}^{n-1} b_i^n(x) \int_0^1 t^k b_{i-1}^{n-2}(t) dt + b_n^n(x).$$

We can write

$$t^k b_{i-1}^{n-2}(t) = \frac{\binom{n-2}{i-1}}{\binom{n+k-2}{k+i-1}} b_{k+i-1}^{n+k-2}(t),$$

and using the well known result [6] $\int_0^1 b_i^n(t) dt = 1/(n+1)$, we obtain

$$\begin{aligned} U_n(t^k, x) &= \frac{(n-1)!(k-1)!}{(n+k-1)!} \sum_{i=1}^{n-1} k \binom{k+i-1}{k} b_i^n(x) + b_n^n(x) \\ &= \frac{(n-1)!(k-1)!}{(n+k-1)!} \sum_{i=0}^{n} k \binom{k+i-1}{k} b_i^n(x). \end{aligned}$$

From the identity (4), we get

$$U_n(t^k, x) = \frac{(n-1)!(k-1)!}{(n+k-1)!} \sum_{i=0}^{n} q \binom{q+i-1}{q} b_i^p(x),$$

and

$$\alpha_i = q \binom{q+i-1}{q},$$

with $q = \max(n, k)$ and $p = \min(n, k)$. The proof is complete.

We consider the Hilbert space $L^2[0, 1]$ with the inner product

$$\langle f, g \rangle_{(\alpha,\beta)} = \int_0^1 f(t) g(t) \omega_{(\alpha,\beta)}(t) dt,$$

where $\omega_{(\alpha,\beta)}(t) = t^\alpha (1-t)^\beta$. from [7, Theorem 4], the operator U_n has eigenvalues

$$\lambda_{n,m} = \frac{(n-1)!n!}{(n-1+m)!(n-m)!}, \ m = 0, 1, ..., n,$$

and for $m \geq 2$ corresponding eigenfunctions $F_{m-2}(x)$ where

$$F_m(x) = \frac{d^m}{dx^m} x^{m+1} (1-x)^{m+1}. \tag{10}$$

Since $\lambda_{n,0} = \lambda_{n,1} = 1$, we see by (2) and (5) that the corresponding eigenfunctions are 1 and x. Notice that the eigenfunctions $F_m(x)$ do not depend on n.

PROPOSITION 2. (i) The eigenfunctions $F_m(x)$ are orthogonal polynomials in $L^2[0,1]$ with respect to the inner product $\langle f, g \rangle_{(-1,-1)}$ and we have

$$\int_0^1 f_m(t) f_j(t) \omega_{(-1,-1)}(t) dt = c_m \delta_{mj},$$

where δ_{mj} is the Kronecker symbol and $c_m = (m!)^2(m+1)/(2m+3)(m+2)$.

(ii) The following recursion formula holds:

$$(n+1)(n+3)F_{n+1}(x) = (2n+3)(n+2)xF_n(x) - (n+1)(n+2)F_{n-1}(x). \tag{11}$$

PROOF. (i). On the interval $[0,1]$ Jacobi polynomials of equal parameters $\alpha = \beta$ can be defined by the Rodrigues formula [1] (upto a constant factor).

$$P_n^a(x) = (x - x^2)^{-\alpha} \frac{d^n}{dx^n}(x - x^2)^{n+\alpha}, \quad x \in [0,1], \tag{12}$$

and the polynomials $P_n(x)$ are orthogonal in $L^2[0,1]$ with respect to the inner product $\langle f, g \rangle_{(\alpha,\alpha)}$. From (12) we obtain

$$F_n(x) = x(1-x)P_n^1(x). \tag{13}$$

Then, $x(1-x)P_n^1(x)P_m^1(x) = x^{-1}(1-x)^{-1}F_n(x)F_m(x)$, which gives

$$\langle P_n(x), P_m(x) \rangle_{(1,1)} = \langle F_n(x), F_m(x) \rangle_{(-1,-1)} = 0, \quad m \neq n.$$

For $m = n$, a repeated integration by parts yields $\|F_m\|^2_{L^2[0,1]} = c_m$.

(ii). Jacobi orthogonal polynomials P_n^1 satisfy the three-term recursion formula

$$(n+1)(n+3)P_{n+1}^1(x) = (2n+3)(n+2)xP_n^1(x) - (n+1)(n+2)P_{n-1}^1(x). \tag{14}$$

From (13) and (14) we obtain the recursion formula (11).

PROPOSITION 3. (i). Let $G_0 = \left\{ f \in L^1[0,1] : f(0) = f(1) = 0 \right\}$. Then for $f, g \in G_0$, we have

$$\langle U_n(f), g \rangle_{(-1,-1)} = \langle f, U_n(g) \rangle_{(-1,-1)},$$

i.e. the operator U_n is self adjoint in G_0 and $\lambda_{n,m} = 0$, $m > n$.

(ii). For any $f \in G_0$, the operator U_n can be expressed in term of $F_m(x)$ as

$$U_n(f, x) = \sum_{m=0}^{n} \lambda_{n,m+2} a_{n,m}(f) F_m(x), \tag{15}$$

where $a_{n,m}(f)$ are the Fourier coefficients of f given by

$$a_{n,m}(f) = \frac{1}{c_m} \langle f, F_m \rangle_{(-1,-1)}.$$

PROOF. (i). Let $f, g \in G_0$. Then

$$\langle U_n(f), g\rangle_{(-1,-1)} = (n-1)\int_0^1 \left(\sum_{i=1}^{n-1} b_i^n(x)(x)\int_0^1 b_{i-1}^{n-2}(t)f(t)dt\right) g(x)\omega_{(-1,-1)}(x)dx.$$

On the other hand we have $b_i^n(x)\omega_{(-1,-1)}(x) = \frac{(n-1)n}{i(n-i)} b_{i-1}^{n-2}$, hence

$$\begin{aligned}\langle U_n(f), g\rangle_{(-1,-1)} &= (n-1)\int_0^1 \left(\sum_{i=1}^{n-1} b_i^n(x)(x)\int_0^1 b_{i-1}^{n-2}(t)g(t)dt\right) f(x)\omega_{(-1,-1)}(x)dx \\ &= \langle f, U_n(g)\rangle_{(-1,-1)}.\end{aligned}$$

Furthermore we have $\lambda_{n,m} = 0$ if $m > n$. Indeed if $r \leq n$ and $m > n$ we have

$$\langle U_n(F_m), F_r\rangle_{(-1,-1)} = \langle F_m, F_r\rangle_{(-1,-1)} \lambda_{n,r} = 0,$$

now by Proposition 1, $U_n(F_m)$ is a degree n polynomial orthogonal to all polynomials of degree $\leq n$, then $U_n(F_m) = 0$ and $\lambda_{n,m} = 0$, $m > n$.

(ii). For integrable function f on $[0,1]$, $U_n(f,x)$ is a polynomial of degree n. Thus there are reals $\alpha_{n,m}(f)$ for $0 \leq m \leq n$ such that

$$U_n(f,x) = \sum_{m=0}^{n} \alpha_{n,m}(f)F_m(x).$$

For $r \leq n$ we have, since U_n is self adjoint

$$\begin{aligned}\langle U_n(f), F_r\rangle_{(-1,-1)} &= \sum_{m=0}^{n} \alpha_{n,m}(f)\langle F_m, F_r\rangle_{(-,-1)} = c_r\alpha_{n,r}(f) \\ &= \langle f, U_n(F_r)\rangle_{(-1,-1)} = \lambda_{n,r+2}\langle f, F_r\rangle_{(-1,-1)},\end{aligned}$$

thus

$$\alpha_{n,r}(f) = \frac{1}{c_r}\lambda_{n,r+2}\langle f, F_r\rangle_{(-1,-1)}.$$

and the proof is complete.

3 Application

The expression (15) of $U_n(f,x)$ is suitable for computation, we shall use this expression for the approximation of given points $q_0, ..., q_r$ in the affine space $\mathbf{R}^d$ by the Bézier curve $U_n(f,x)$. We can always assume that $q_0 = q_r = 0$. Let $0 = t_0 < t_1 < \cdots < t_{r-1} < t_r = 1$ be a subdivision of the interval $[0,1]$ and f the piecewise affine function such that $f(t_i) = q_i$, $i = 0, ..., r$, defined on $[0,1]$ by

$$f(u) = ((t_{i+1-ti})q_i + (u-t_i)q_{i+1})/(t_{i+1}-t_i), \quad u \in [t_i, t_{i+1}].$$

PROPOSITION 4. Let d be a distance in $\mathbf{R}^d$ associated with a norm denoted $\|\cdot\|$.
(i). For any integer n, one has the estimate for $i = 1, ..., r$,

$$d(U_n(f, t_i), q_i) = 2\Delta n^{-1/2} \tag{16}$$

where $\Delta = \max_{0 \le h \le r} \|\Delta_h\|$ and $\Delta_h = (q_{h+1} - q_h)/(t_{h+1} - t_h)$.

(ii). Let $A_k = \langle f, F_k \rangle_{(-1,-1)}$. Then $U_n(f, x) = \sum_{k=0}^{n} \frac{\lambda_{n,k+2}}{c_k} A_k F_k(x)$ with

$$A_k = \frac{-1}{(k+2)(k+1)} \sum_{i=1}^{r-1} F_k(t_i)\Delta_i^2, \quad k = 0, ..., r, \tag{17}$$

where $\Delta_i^2 = \Delta_i - \Delta_{i-1}$.

PROOF. (i). Inequality (16) is a consequence of the next estimation given by [7, Theorem 11]

$$\|U_n(f) - f_k\| = 2\omega(f; n^{-1/2}),$$

where $\omega(f; h) = \sup_{(x,t)\in H} |f(x + t) - f(x)|$, $H = \left\{(x, t), / |t| = h, (x + t, x) \in [0, 1]^2\right\}$, is the moduli of continuity of f.

(ii). It is known [7] that $F_m(x)$ satisfies the differential equation

$$x(1 - x)y_{xx} + (m + 2)(m + 1)y = 0,$$

and it follows immediately that

$$\begin{aligned} A_k &= e_k \int_0^1 f(u)F_k''(u)du = e_k \sum_{i=0}^{r-1} \int_{t_i}^{t_{i+1}} f(u)F''k(u)du \\ &= e_k \left(q_r F_m'(1) - q_0 F_m'(0) - \Delta_{r-1}F_m(1) + \Delta_0 F_m(0) + \sum_{i=1}^{r-1} F_m(t_i)\Delta_i^2 \right), \end{aligned}$$

with $e_k = \frac{-1}{(k+2)(k+1)}$. Since $q_0 = q_r = 0$ and $F_m(1) = F_m(0) = 0$, we obtain (17).

The three-term recursion formula (11) provides an efficient and fast algorithm for computing the values of $U_n(f)$ expressed in term of the polynomials $F_m(x)$ with only $O(n)$ multiplications while the de Casteljau algorithm [6] related to Bernstein basis polynomials needs $O(n^2)$ multiplications.

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