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EIGENVALUES OF SEVERAL TRIDIAGONAL MATRICES *

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Abstract

Tridiagonal matrices appear frequently in mathematical models. In this note, we derive the eigenvalues and the corresponding eigenvectors of several tridiagonal matrices by the method of symbolic calculus in [1].

1 Introduction

There are many mathematical models which involves tridiagonal matrices of the form [2]

$$A_n = \begin{pmatrix} -\alpha+b & c & 0 & 0 & \dots & 0 & 0 \\ a & b & c & 0 & \dots & 0 & 0 \\ 0 & a & b & c & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & a & -\beta+b \end{pmatrix}_{n\times n} . \tag{1}$$

In particular, when $a = c = 1$, $b = -2$ and $\alpha = \beta = 0$, the eigenvalues of A_n has been proved [3,4] to be

$$\lambda_k(A_n) = -2 + 2\cos\frac{k\pi}{n+1},\ k = 1, 2, ..., n;$$

when $a = c = 1$, $b = -2$ and $\alpha = \beta = 1$, or, when $a = c = 1$, $b = -2$, $\alpha = 1$ and $\beta = 0$, the eigenvalues have been reported as

$$\lambda_k(A_n) = -2 + 2\cos\frac{k\pi}{n},\ k = 1, 2, ..., n;$$

or

$$\lambda_k(A_n) = -2 + 2\cos\frac{2k\pi}{2n+1},\ k = 1, 2, ..., n$$

respectively without proof. In this note, we intend to derive the eigenvalues and the corresponding eigenvectors of several tridiagonal matrices of the form A_n.

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2 The Eigenvalue Problem

Consider the eigenvalue problem $A_n u = \lambda u$, where a, b, c and α, β are numbers in the complex plane C. We will assume that $ac \neq 0$ since the contrary case is easy.

Let λ be an eigenvalue (which may be complex) and $(u_1, ..., u_n)^\dagger$ a corresponding eigenvector. We may view the numbers $u_1, u_2, ..., u_n$ respectively as the first, second, ..., and the n-th term of an infinite (complex) sequence $u = \{u_i\}_{i=0}^\infty$. Since $A_n u = \lambda u$ can be written as

$$\begin{aligned}
u_0 &= 0 \\
au_0 + bu_1 + cu_2 &= \lambda u_1 + \alpha u_1, \\
au_1 + bu_2 + cu_3 &= \lambda u_2 + 0, \\
... &= ..., \\
au_{n-2} + bu_{n-1} + cu_n &= \lambda u_{n-1} + 0, \\
au_{n-1} + bu_n + cu_{n+1} &= \lambda u_n + \beta u_n, \\
u_{n+1} &= 0,
\end{aligned}$$

we see that the sequence $u = \{u_k\}_{k=0}^\infty$ satisfies $u_0 = 0, u_{n+1} = 0$ and

$$au_{k-1} + bu_k + cu_{k+1} = \lambda u_k + f_k,\ \ k = 1, 2, ..., \tag{2}$$

where $f_1 = \alpha u_1$ and $f_n = \beta u_n$, while $f_k = 0$ for $k \neq 1, n$. Note that u_1 cannot be 0, for otherwise from (2), $cu_2 = 0$ and inductively $u_3 = u_4 = \cdots = u_n = 0$ which is contrary to the definition of an eigenvector.

Let $f = \{f_k\}_{k=0}^\infty$ be defined above. Then (2) can be expressed as

$$c\{u_{k+2}\}_{k=0}^\infty + b\{u_{k+1}\}_{k=0}^\infty + a\{u_k\}_{k=0}^\infty = \lambda\{u_{k+1}\}_{k=0}^\infty + \{f_{k+1}\}_{k=0}^\infty.$$

We now recall that $\hbar = \{0, 1, 0, ...\}$, $\overline{\alpha} = \{\alpha, 0, 0, ...\}$ and the properties of convolution product xy of two sequences $x = \{x_k\}_{k=0}^\infty$ and $y = \{y_k\}_{k=0}^\infty$ (see [1] for details). Then by taking convolution of the above equation with $\hbar^2 = \hbar\hbar$, and noting that

$$\hbar\{u_{n+1}\} = \hbar\{u_1, u_2, ...\} = \{0, u_1, u_2, ...\} = u - \overline{u_0}$$

and

$$\hbar^2\{u_{n+2}\} = \hbar^2\{u_2, u_3, ...\} = \{0, 0, u_2, u_3, ...\} = u - \overline{u_0} - u_1\hbar,$$

we have

$$c(u - \overline{u_0} - u_1\hbar) + (b - \lambda)\hbar(u - \overline{u_0}) + a\hbar^2 u = \hbar\left(f - \overline{f_0}\right).$$

Solving for u, and substituting $u_0 = f_0 = 0$, we obtain

$$\left(a\hbar^2 + (b - \lambda)\hbar + \overline{c}\right)u = (f + c\overline{u_1})\hbar.$$

Since $c \neq 0$, we can divide the above equation [1] by $a\hbar^2 + (b - \lambda)\hbar + \overline{c}$ to obtain

$$u = \frac{(f + c\overline{u_1})\hbar}{a\hbar^2 + (b - \lambda)\hbar + \overline{c}}. \tag{3}$$

Let

$$\gamma_{\pm} = \frac{-(b-\lambda) \pm \sqrt{\omega}}{2a},\ ac \neq 0$$

be the two roots of $az^2 + (b-\lambda)z + c = 0$, where $\omega = (b-\lambda)^2 - 4ac$. Since a, b, c as well as $\gamma_{\pm}, \omega$ are in the complex domain, we first introduce the following Lemma.

LEMMA 1. Let $z = x + iy$ where $z \in C$ and $x, y \in R$. Then (i) $\sin z = 0$ if and only if $z = x = k\pi$ for some $k \in Z$, and (ii) $\cos z = \pm 1$ if and only if $z = x = j\pi$ for some $j \in Z$.

PROOF. If $z = x = k\pi$, $k \in Z$, then $\sin z = 0$, which gives the sufficient condition of (i). If

$$\sin z = \sin(x+iy) = \sin x \cosh y + i(\cos x \sinh y) = 0,$$

then $\sin x \cosh y = 0$ and $\cos x \sinh y = 0$. Since $\cosh y \neq 0$, hence $\sin x = 0$ so that $x = k\pi$, $k \in Z$. Consequently $\cos x \neq 0$ and $\sinh y = 0$, which yields $y = 0$. Hence $z = x = k\pi$, $k \in Z$. This gives the necessary condition of (i). To prove (ii), in a similar manner we see that if $z = k\pi$, $k \in Z$, than $\cos z = \pm 1$. On the other hand, if

$$\cos z = \cos(x+iy) = \cos x \cosh y - i(\sin x \sinh y) = \pm 1,$$

then $\cos x \cosh y = \pm 1$ and $\sin x \sinh y = 0$. If $\sin x \neq 0$, then $\sinh y = 0$ so that $y = 0$, consequently $\cos x = \pm 1$ and $x = k\pi$, $k \in Z$. But then $\sin x = 0$ which contradicts the assumption $\sin x \neq 0$. Hence $\sin x = 0$ and $x = k\pi$, $k \in Z$. Then $\cos x = \pm 1$ and $\cosh y = 1$, which demands $y = 0$. This completes the proof.

COROLLARY 1. If $z \neq k\pi$ where $k \in Z$, then $\sin z \neq 0$, $\cos z \neq \pm 1$ and $\sin \frac{z}{2} \neq 0$, $\cos \frac{z}{2} \neq 0$.

PROOF. If $z \neq k\pi$, $\sin z \neq 0$ and $\cos z \neq \pm 1$ follows readily from Lemma 1. Since $\sin z = 2 \sin \frac{z}{2} \cos \frac{z}{2} \neq 0$, so we have $\sin \frac{z}{2} \neq 0$ and $\cos \frac{z}{2} \neq 0$. This completes the proof.

According to $\gamma_{\pm}$ being two different complex numbers or two equal numbers, there are two cases to be considered.

Case I. Suppose γ_+ and γ_- are two different complex numbers. Let $\gamma_{\pm} = p \pm iq$ where $p, q \in C$ and $q \neq 0$. Since $\gamma_+\gamma_- = p^2 + q^2 = c/a$ and $\gamma_+ + \gamma_- = 2p = (\lambda - b)/a$, we may write

$$\gamma_{\pm} = \sqrt{p^2+q^2}(\cos\theta \pm i \sin\theta) = \frac{1}{\rho} e^{\pm i\theta},$$

where

$$\rho = \sqrt{\frac{a}{c}},\ \cos\theta = \frac{p}{\sqrt{p^2+q^2}} = \frac{\lambda - b}{2\sqrt{ac}},\ \rho, \theta \in C. \tag{4}$$

By the method of partial fractions,

$$\begin{aligned} u &= \frac{1}{\sqrt{\omega}} \left(\frac{1}{\gamma_- - \hbar} - \frac{1}{\gamma_+ - \hbar} \right) (f + c\overline{u_1})\,\hbar \\ &= \frac{1}{\sqrt{\omega}} \left\{ \gamma_-^{-(j+1)} - \gamma_+^{-(j+1)} \right\} (f + c\overline{u_1})\,\hbar \\ &= \frac{1}{\sqrt{\omega}} \left\{ \left(\frac{a}{c}\right)^{j+1} \left(\gamma_+^{j+1} - \gamma_-^{j+1} \right) \right\} (f + c\overline{u_1})\,\hbar, \end{aligned}$$

where the last two equalities are due to $\frac{1}{a-\hbar}=\left\{a^{-(n+1)}\right\}_{n=0}^{\infty}$ and $\gamma_+\gamma_-=c/a$. Applying De Moivre's Theorem, this may further be written as

$$u=\frac{2i}{\sqrt{\omega}}\left\{\rho^{j+1}\sin\left(j+1\right)\theta\right\}\left(f+c\overline{u_1}\right)\hbar.$$

Setting $f_1=\alpha u_1$, $f_n=\beta u_n$ and $f_j=0$ for $j\neq 1,n$, we may evaluate u_j and obtain

$$u_j=\frac{2i}{\sqrt{\omega}}\left(cu_1\rho^j\sin j\theta+\alpha u_1\rho^{j-1}\sin\left(j-1\right)\theta+H\left(j-n-1\right)\beta u_n\rho^{j-n}\sin\left(j-n\right)\theta\right) \tag{5}$$

for $j\geq 1$, where $H\left(x\right)$ is the unit step function defined by $H(x)=1$ if $x\geq 0$ and $H(x)=0$ if $x<0$. In particular,

$$\begin{aligned}\frac{\sqrt{\omega}}{2i}u_{n+1} &= cu_1\rho^{n+1}\sin\left(n+1\right)\theta+\alpha u_1\rho^n\sin n\theta+\beta u_n\rho\sin\theta\\ &= cu_1\rho^{n+1}\sin\left(n+1\right)\theta+\alpha u_1\rho^n\sin n\theta\\ &\quad+\beta\rho\frac{2i\sin\theta}{\sqrt{\omega}}\left(cu_1\rho^n\sin n\theta+\alpha u_1\rho^{n-1}\sin\left(n-1\right)\theta\right)\\ &= cu_1\rho^{n+1}\sin\left(n+1\right)\theta+\left(\alpha+\beta\right)u_1\rho^n\sin n\theta+\alpha\beta u_1\frac{1}{c}\rho^{n-1}\sin\left(n-1\right)\theta,\end{aligned}$$

where we have substituted $2i\sqrt{ac}\sin\theta=\sqrt{\omega}$. Since $\rho,u_1\neq 0$ and $u_{n+1}=0$, we finally obtain the necessary condition

$$ac\sin\left(n+1\right)\theta+\left(\alpha+\beta\right)\sqrt{ac}\sin n\theta+\alpha\beta\sin\left(n-1\right)\theta=0. \tag{6}$$

Since $\gamma_+\neq\gamma_-$, $\gamma_+-\gamma_-=2i\sqrt{\frac{c}{a}}\sin\theta\neq 0$. By Lemma 1, $\theta\neq m\pi$ for $m\in Z$. Then by (4), we have

$$\lambda=b+2\sqrt{ac}\cos\theta\ \theta\neq m\pi,\quad m\in Z. \tag{7}$$

Note that we may also obtain from (5) that

$$\begin{aligned}u_j &= \frac{2i}{\sqrt{\omega}}\left(cu_1\rho^j\sin j\theta+\alpha u_1\rho^{j-1}\sin\left(j-1\right)\theta\right)\\ &= \frac{u_1}{\sin\theta}\rho^{j-1}\left(\sin j\theta+\frac{\alpha}{\sqrt{ac}}\sin\left(j-1\right)\theta\right)\end{aligned} \tag{8}$$

for $j=1,2,...,n$.

Case II. $\gamma_\pm$ are two equal roots. In this case, $q=0$, or $\omega=\left(b-\lambda\right)^2-4ac=0$. So we have

$$\lambda=b\pm 2\sqrt{ac}. \tag{9}$$

Furthermore, from (3), we have

$$\begin{aligned}u &= \frac{\left(f+c\overline{u_1}\right)\hbar}{c\left(1+\frac{b-\lambda}{c}\hbar+\frac{a}{c}\hbar^2\right)}=\frac{\left(f+c\overline{u_1}\right)\hbar}{c\left(1\mp 2\sqrt{\frac{a}{c}}\hbar+\left(\sqrt{\frac{a}{c}}\hbar\right)^2\right)}\\ &= \frac{1}{\sqrt{ac}}\frac{\rho\hbar}{\left(1\mp\rho\hbar\right)^2}\left(f+c\overline{u_1}\right)=\frac{1}{\sqrt{ac}}\left\{\left(\pm 1\right)^{j+1}j\rho^j\right\}\left(f+c\overline{u_1}\right)\end{aligned}$$

The j-th term now becomes

$$\begin{aligned} u_j &= \frac{1}{\sqrt{ac}}\left((\pm 1)^{j+1} cu_1 j\rho^j + (\pm 1)^j \alpha u_1 (j-1)\rho^{j-1}\right) \\ &\quad + \frac{1}{\sqrt{ac}} (\pm 1)^{j-n+1} H(j-n-1)\beta u_n (j-n) \rho^{j-n}. \end{aligned} \tag{10}$$

By letting $u_{n+1} = 0$, we obtain

$$\left(ac \mp (\alpha+\beta)\sqrt{ac} + \alpha\beta\right) n + (ac - \alpha\beta) = 0.$$

Since this formula must be valid for all $n \geq 2$, thus $ac \pm (\alpha+\beta)\sqrt{ac} + \alpha\beta = 0$ and $ac - \alpha\beta = 0$. This yields the necessary condition $\alpha = \beta = \mp\sqrt{ac}$ (where $\alpha = \beta = -\sqrt{ac}$ corresponds to the eigenvalue $\lambda = b + 2\sqrt{ac}$, and $\alpha = \beta = \sqrt{ac}$ corresponds to the eigenvalue $\lambda = b - 2\sqrt{ac}$). The corresponding eigenvectors may be obtained from (10). Since $j \leq n$, we have, if we set $u_1 = 1$, $u_j = (-\rho)^{j-1}$ when $\alpha = \sqrt{ac}$ and $u_j = \rho^{j-1}$ when $\alpha = -\sqrt{ac}$.

3 Special Tridiagonal Matrices

Now we can apply the results of the last section to find the eigenvalues of several tridiagonal matrices of the form (1). We will assume $ac \neq 0$ and set $\rho = \sqrt{a/c}$ as before.

Suppose $\alpha = \beta = 0$ in A_n. Suppose λ is an eigenvalue. In Case I, (6) reduces to

$$\sin(n+1)\theta = 0.$$

Hence by Lemma 1,

$$\theta = \frac{k\pi}{n+1},\ k = 0, \pm 1, \pm 2, \ldots.$$

Case II does not hold since $0 = \alpha = \beta = \sqrt{ac}$ is not allowed.

In other words, if λ is an eigenvalue of A_n and $(u_1, u_2, \ldots, u_n)^\dagger$ is a corresponding eigenvector, then according to (7),

$$\lambda = b + 2\sqrt{ac}\cos\frac{k\pi}{n+1}$$

for some $k \in \{1, \ldots, n\}$, and the corresponding $u_j^{(k)}$, according to (8), is given by

$$u_j^{(k)} = \rho^{j-1}\sin\frac{kj\pi}{n+1},\ j = 1, 2, \ldots, n. \tag{11}$$

where we have assumed $u_1^{(k)} = \sin\frac{k\pi}{n+1}$.

Conversely, we may check by reversing the arguments in Section 2 that for each $k \in \{1, \ldots, n\}$, the number

$$\lambda_k = b + 2\sqrt{ac}\cos\frac{k\pi}{n+1},\ k = 1, 2, \ldots, n, \tag{12}$$

is an eigenvalue and the vector $u^{(k)} = (u_1^{(k)}, u_2^{(k)}, ..., u_n^{(k)})^\dagger$ a corresponding eigenvector of A_n.

Before proceeding further, we introduce the following Lemma.

LEMMA 2. Let

$$B_n = \begin{pmatrix} -\beta + b & c & 0 & 0 & ... & 0 & 0 \\ a & b & c & 0 & ... & 0 & 0 \\ 0 & a & b & c & ... & 0 & 0 \\ ... & ... & ... & ... & ... & \cdots & \cdots \\ 0 & 0 & 0 & 0 & ... & a & -\alpha + b \end{pmatrix}_{n\times n},$$

which is obtained from A_n by interchanging the numbers α and β. Then the eigenvalues of B_n are the same as A_n, and the corresponding eigenvectors $v^{(k)} = \left(v_1^{(k)}, ..., v_n^{(k)}\right)^\dagger$, $k = 1, ..., n$, are given by

$$v_j^{(k)} = \rho^{2j} u_{n-j+1}^{(k)}, \ k = 1, 2, ..., n \tag{13}$$

where $u^{(k)} = \left(u_1^{(k)}, ..., u_n^{(k)}\right)^\dagger$, $k = 1, ..., n$, are the eigenvectors of A_n.

PROOF. Let λ be an eigenvalue and $u = (u_1, ..., u_n)^\dagger$ a corresponding eigenvector of A_n. Let

$$R_n = \begin{pmatrix} 0 & 0 & ... & 0 & \rho^2 \\ 0 & 0 & ... & \rho^4 & 0 \\ ... & ... & ... & ... & ... \\ 0 & \rho^{2n-2} & ... & 0 & 0 \\ \rho^{2n} & 0 & ... & 0 & 0 \end{pmatrix}_{n\times n}.$$

Then since $A_n u = \lambda u$, we have $R_n A_n R_n^{-1} R_n u = \lambda R_n u$ or $A_n^* u^* = \lambda u^*$, where $u^* = R_n u = \left(\rho^2 u_n, \rho^4 u_{n-1}, ..., \rho^{2n} u_1\right)^\dagger$ and $A_n^* = R_n A_n R_n^{-1}$. By noting that $\rho^2 c = a$ and $\rho^{-2} a = c$, it is not difficult to see that $A_n^* = B_n$. Let $v = u^*$, then we have $B_n v = \lambda v$. Thus B_n has the same eigenvalues λ as A_n, and the corresponding eigenvectors $v = (v_1, ..., v_n)^\dagger$ are given by $v_j = \rho^{2j} u_{n-j+1}$. This completes the proof.

Now suppose $\alpha = 0$ and $\beta = \sqrt{ac} \neq 0$. This yields $\alpha\beta = 0$ and $\alpha + \beta = \sqrt{ac}$. In Case I, (6) becomes

$$\sin(n+1)\theta + \sin n\theta = 0.$$

or

$$2 \sin \frac{(2n+1)\theta}{2} \cos \frac{\theta}{2} = 0.$$

Since $\theta \neq m\pi$, $m \in Z$, by Corollary of Lemma 1, $\cos\frac{\theta}{2} \neq 0$, we have $\sin\frac{(2n+1)\theta}{2} = 0$. Thus

$$\theta = \frac{2k\pi}{2n+1}, \ k = 0, \pm 1, \pm 2, ...$$

Case II does not hold since $\alpha = 0 \neq \sqrt{ac}$. By reasons similar to the case where $\alpha = \beta = 0$ above, we may now see the following.

THEOREM 1. Suppose $\alpha = 0$ and $\beta = \sqrt{ac} \neq 0$. Then the eigenvalues $\lambda_1, ..., \lambda_n$ of A_n are given by

$$\lambda_k = b + 2\sqrt{ac}\cos\frac{2k\pi}{2n+1},\ k = 1, 2, ..., n. \tag{14}$$

The corresponding eigenvectors $u^{(k)} = \left(u_1^{(k)}, ..., u_n^{(k)}\right)^\dagger$, $k = 1, ..., n$ are given by

$$u_j^{(k)} = \rho^{j-1}\sin\frac{2kj\pi}{2n+1},\ j = 1, 2, ..., n.$$

We remark that in case $\beta = 0$ and $\alpha = \sqrt{ac} \neq 0$, Lemma 2 says that the eigenvalues are the same as given by (14). The corresponding eigenvector $v^{(k)} = \left(v_1^{(k)}, ..., v_n^{(k)}\right)^\dagger$, $k = 1, ..., n$, in view of (13), are

$$v_j^{(k)} = \rho^{j-1}\sin\frac{k(2j-1)\pi}{2n+1},\ j = 1, 2, ..., n. \tag{15}$$

The eigenvalues and the corresponding eigenvectors of the other case $\alpha\beta = 0$ and $\alpha + \beta = -\sqrt{ac}$ can be obtained in a similar way. In Case I, now (6) becomes $\sin(n+1)\theta - \sin n\theta = 0$ or

$$2\cos\frac{(2n+1)\theta}{2}\sin\frac{\theta}{2} = 0.$$

Since $\theta \neq m\pi$, $m \in Z$, by Corollary of Lemma 1, $\sin\frac{\theta}{2} \neq 0$, we have $\cos\frac{(2n+1)\theta}{2} = 0$. Thus

$$\theta = \frac{(2k-1)\pi}{2n+1},\ k = \pm1, \pm2, \pm3, ...$$

THEOREM 2. Suppose $\alpha = 0$ and $\beta = -\sqrt{ac} \neq 0$. Then the eigenvalues $\lambda_1, ..., \lambda_n$ of A_n are given by

$$\lambda_k = b + 2\sqrt{ac}\cos\frac{(2k-1)\pi}{2n+1},\ k = 1, 2, 3, ..., n. \tag{16}$$

The corresponding eigenvectors $u^{(k)} = \left(u_1^{(k)}, ..., u_n^{(k)}\right)^\dagger$, $k = 1, ..., n$, are given by

$$u_j^{(k)} = \rho^{j-1}\sin\frac{(2k-1)j\pi}{2n+1},\ j = 1, 2, ..., n.$$

In case $\beta = 0$ and $\alpha = -\sqrt{ac} \neq 0$, the eigenvalues are given by (16) and the corresponding eigenvectors by

$$v_j^{(k)} = \rho^{j-1}\cos\frac{(2k-1)(2j-1)\pi}{2(2n+1)},\ j = 1, 2, ..., n.$$

Next, suppose $\alpha = -\beta = \sqrt{ac} \neq 0$, then (6) reduces to

$$\sin(n+1)\theta - \sin(n-1)\theta = 0$$

or

$$2 \cos n\theta \sin \theta = 0.$$

Since $\sin \theta \neq 0$, thus $\cos n\theta = 0$, so that

$$\theta = \frac{(2k-1)\,\pi}{2n},\ k = \pm 1, \pm 2, \pm 3, ...$$

Cases II does not hold as before.

THEOREM 3. Suppose $\alpha = -\beta = \sqrt{ac} \neq 0$. Then the eigenvalues $\lambda_1, ..., \lambda_n$ of A_n are given by

$$\lambda_k = b + 2\sqrt{ac} \cos \frac{(2k-1)\,\pi}{2n},\ k = 1, 2, 3, ..., n. \tag{17}$$

The corresponding eigenvectors $u^{(k)} = \left(u_1^{(k)}, ..., u_n^{(k)}\right)^{\dagger}$, $k = 1, ..., n$, according to (8), are given by

$$u_j^{(k)} = \rho^{j-1} \sin \frac{(2k-1)\,(2j-1)\,\pi}{4n},\ j = 1, 2, ..., n.$$

In case $\alpha = -\beta = -\sqrt{ac} \neq 0$, the eigenvalues are given by (17) and the corresponding eigenvectors by

$$v_j^{(k)} = \rho^{j-1} \cos \frac{(2k-1)\,(2j-1)\,\pi}{4n},\ j = 1, 2, ..., n.$$

Next, suppose $\alpha = \beta = \sqrt{ac} \neq 0$, or $\alpha = \beta = -\sqrt{ac} \neq 0$. If λ is an eigenvalue of A_n, then in Case I, (6) reduces to

$$2 \sin n\theta \,(\cos \theta + 1) = 0.$$

or

$$2 \sin n\theta \,(\cos \theta - 1) = 0$$

respectively. Since $\theta \neq m\pi$, $m \in Z$, by Corollary of Lemma 1, $\cos \theta \pm 1 \neq 0$, we have $\sin n\theta = 0$. Thus

$$\theta = \frac{k\pi}{n},\ k = 0, \pm 1, \pm 2, \pm 3, ...\ .$$

Since $\theta \neq m\pi$ for $m \in Z$ and since $\cos \theta$ is even and periodic, we obtain

$$\lambda = b + 2\sqrt{ac} \cos \frac{k\pi}{n},\ k = 1, 2, 3, ..., n-1.$$

In Case II, by (9), we have $\lambda = b + 2\sqrt{ac} = b + 2\sqrt{ac} \cos 0$ if $\alpha = \beta = -\sqrt{ac}$, and $\lambda = b - 2\sqrt{ac} = b + 2\sqrt{ac} \cos \pi$ if $\alpha = \beta = \sqrt{ac}$.

THEOREM 4. Suppose $\alpha = \beta = \sqrt{ac} \neq 0$. Then the eigenvalues $\lambda_1, ..., \lambda_n$ of A_n are given by

$$\lambda_k = b + 2\sqrt{ac} \cos \frac{k\pi}{n},\ k = 1, 2, 3, ..., n,$$

and the corresponding eigenvectors $u^{(k)} = \left(u_1^{(k)}, ..., u_n^{(k)}\right)^\dagger$ are given by

$$u_j^{(k)} = \rho^{j-1} \sin \frac{k(2j-1)\pi}{2n},\ j = 1, 2, ..., n,$$

for $k = 1, 2, ..., n-1$ and

$$u_j^{(k)} = (-\rho)^{j-1},\ j = 1, 2, ..., n,$$

for $k = n$.

THEOREM 5. Suppose $\alpha = \beta = -\sqrt{ac} \neq 0$. Then the eigenvalues $\lambda_1, ..., \lambda_n$ of A_n are given by

$$\lambda_k = b + 2\sqrt{ac} \cos \frac{(k-1)\pi}{n},\ k = 1, 2, 3, ..., n,$$

and the corresponding eigenvectors $u^{(k)} = \left(u_1^{(k)}, ..., u_n^{(k)}\right)^\dagger$ are given by

$$u_j^{(k)} = \rho^{j-1},\ j = 1, 2, ..., n$$

for $k = 1$ and

$$u_j^{(k)} = \rho^{j-1} \cos \frac{(k-1)(2j-1)\pi}{2n},\ j = 1, 2, ..., n,$$

for $k = 2, 3, ..., n$.

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