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ON THE POLAR MOMENT OF INERTIA OF THE PROJECTION CURVE*

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Abstract

The polar moment of inertia for the orbit curves during the 1-parameter closed planar motions is given by H. R. Müller [2]. In this paper, under the 1-parameter closed motion in three dimensional Euclidean space, we expressed the polar moment of inertia of the projection curve of the closed space curve. We also obtained formulas equivalent to the results given by [2] and [3].

1 Introduction

A general displacement in Euclidean n-space is given analytically by

$$\mathbf{x}' = A\mathbf{x} + C \tag{1}$$

in which $\mathbf{x}'$ and $\mathbf{x}$ are the position vectors, represented by column matrices, of a point X in the fixed space R' and the moving space R respectively; A is an orthogonal matrix and C a translation vector. If A and C are functions of a parameter t, which may be identified with time, (1) gives us a continuous series of displacements, called a *motion*. Without any loss of generality we may suppose that for $t = 0$ the origins in R and R' coincide, so for $t = 0$, $A = I$ and $C = 0$.

In this study, we consider the 1-parameter closed motion in Euclidean 3-space. Since the motion is closed, all the quantities depending on the parameter t are the periodic functions of the same period T. Let $\{O; \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ and $\{O'; \mathbf{e_1}', \mathbf{e_2}', \mathbf{e_3}'\}$ be two right-handed sets of orthonormal vectors that are rigidly linked to the moving space R and fixed space R', respectively, and denote E, E' the matrices

$$E = \begin{bmatrix} \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{e_3} \end{bmatrix}, \quad E' = \begin{bmatrix} \mathbf{e_1}' \\ \mathbf{e_2}' \\ \mathbf{e_3}' \end{bmatrix}.$$

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Then we may write

$$E = AE' \quad \text{or} \quad E' = A^t E, \tag{2}$$

where A is a positive orthogonal 3×3-matrix and the superscript "t" indicates the transpose. Since $A \in SO(3)$ we may write

$$AA^t = I,$$

where I is the unit matrix. This equation, by differentiation with respect to t, yields

$$dA.A^t + A.dA^t = 0,$$

which shows that the matrix

$$\Omega = dA.A^t$$

is antisymmetric. We may write

$$\Omega = \begin{bmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{bmatrix},$$

where $w_i,\ i = 1, 2, 3$, are the linear differential forms with respect to t, i.e. $w_i = f_i(t)dt$. Differentiation of (2) with respect to t yields

$$dE = \Omega E$$

or

$$d\mathbf{e_i} = w_k \mathbf{e_j} - w_j \mathbf{e_k}, \quad (i, j, k = 1, 2, 3; 2, 3, 1; 3, 1, 2). \tag{3}$$

Then, we may write

$$d\mathbf{e_i} = \boldsymbol{\omega} \times \mathbf{e_i},$$

where

$$\boldsymbol{\omega} = w_1 \mathbf{e_1} + w_2 \mathbf{e_2} + w_3 \mathbf{e_3}$$

is called the rotation vector of the motion and " $\times$ " denotes the vector product.

2 The Polar Moment of Inertia of the Projection Curve

I.

Let X be a fixed point in R with

$$\vec{OX} = \mathbf{x} = x_1 \mathbf{e_1} + x_2 \mathbf{e_2} + x_3 \mathbf{e_3}.$$

If we denote

$$\vec{OO'} = \mathbf{u} = u_1\mathbf{e_1} + u_2\mathbf{e_2} + u_3\mathbf{e_3},$$

for the position vector of X in R' we may write

$$\vec{O'X} = \mathbf{x'} = \mathbf{x} - \mathbf{u}. \tag{4}$$

The point X describes a closed curve (X), its *path*, in R' during the 1-parameter closed motion. The projection of this closed path, in the direction of a fixed unit vector $\mathbf{e}'$, on any plane P is a closed curve, say (X^n). We suppose that the curve (X^n) is uniformly covered with the mass elements

$$dm = ||\boldsymbol{\omega}^{\mathbf{n}}||dt = |cos\theta|\omega dt,$$

where $\omega = ||\boldsymbol{\omega}||$ is the instantaneous angular velocity of the motion, $\boldsymbol{\omega}^{\mathbf{n}}$ is the normal component of $\boldsymbol{\omega}$ to the plane P and $\theta = \theta(t)$ is the angle between the vectors $\boldsymbol{\omega}$ and $\mathbf{e}'$.

For the projection of $\mathbf{x}'$ in the direction of $\mathbf{e}'$ on P, we have

$$\mathbf{x^n} = \mathbf{x}' - < \mathbf{e}', \mathbf{x}' > \mathbf{e}', \tag{5}$$

where $\mathbf{x^n}$ is the position vector of the projection point X^n of $X' \in R'$ and " $<,>$ " denotes the scalar product. Thus, the polar moment of inertia of the curve (X^n) with respect to the origin O' of R' $(O' \in P)$ is

$$M_X = \oint ||\mathbf{x^n}||^2 dm, \tag{6}$$

where the integration is taken along the closed curve (X^n).

If we substitute (5) into (6), for the polar moment of inertia of the projection curve of (X) we obtain

$$M_X = \int_0^T \left\{ ||\mathbf{x}'||^2 - (< \mathbf{e}', \mathbf{x}' >)^2 \right\} |cos\theta|\omega dt, \tag{7}$$

where the integration is taken along the closed curve (X).

Let the direction of projection in R' is given by the unit vector $\mathbf{e}' = a_1\mathbf{e_1} + a_2\mathbf{e_2} + a_3\mathbf{e_3}$. Then, if we substitute (4) into (7), we get

$$M_X = M_O + \sigma \sum_{i=1}^{3} x_i^2 - \sum_{i,j=1}^{3} b_{ij}x_ix_j + \sum_{i=1}^{3} c_ix_i, \tag{8}$$

where

$$\sigma = \int_0^T |cos\theta|\omega dt, \quad b_{ij} = \int_0^T a_ia_j|cos\theta|\omega dt, \quad c_i = 2\int_0^T \left\{ a_i \sum_{j=1}^{3} a_ju_j - u_i \right\} |cos\theta|\omega dt$$

and M_O is the polar moment of inertia of the projection curve of the orbit curve (O).

We may give the following theorem.

THEOREM 1. Let us consider the 1-parameter closed motions of Euclidean 3-space. If the projection curves (in the direction of a unit vector $\mathbf{e}'$) of closed point paths have equal polar moment of inertia, then such points lie on the same sphere.

SPECIAL CASE. In the case of $\theta(t) \equiv 0$, the paths are planar. Thus, we obtain the polar moment of inertia given by Müller, [2].

II.

Let X and Y be two fixed points in R and Z be another point on the line segment XY. Then we may write

$$z_i = \lambda x_i + \xi y_i, \quad \lambda + \xi = 1. \tag{9}$$

Using (8), we get

$$M_Z = \lambda^2 M_X + 2\lambda\xi M_{XY} + \xi^2 M_Y, \tag{10}$$

where

$$M_{XY} = M_O + \sigma \sum_{i=1}^{3} x_i y_i + \frac{1}{2}\left\{ \sum_{i=1}^{3} c_i(x_i + y_i) - \sum_{i,j=1}^{3} b_{ij}(x_i y_j + x_j y_i) \right\} \tag{11}$$

is called the *mixture polar moment of inertia* of the projection curves of (X) and (Y). It is clearly seen that $M_{XY} = M_{YX}$ and $M_{XX} = M_X$.

Since

$$M_X - 2M_{XY} + M_Y = \sigma \sum_{i=1}^{3} (x_i - y_i)^2 - \sum_{i,j=1}^{3} b_{ij}(x_i - y_i)(x_j - y_j), \tag{12}$$

we can rewrite (10) as follows:

$$M_Z = \lambda M_X + \xi M_Y - \lambda\xi \left\{ \sigma \sum_{i=1}^{3} (x_i - y_i)^2 - \sum_{i,j=1}^{3} b_{ij}(x_i - y_i)(x_j - y_j) \right\}. \tag{13}$$

We will define the distance $D(X, Y)$ between the points X, Y of R by

$$D^2(X, Y) = \varepsilon \left\{ \sigma \sum_{i=1}^{3} (x_i - y_i)^2 - \sum_{i,j=1}^{3} b_{ij}(x_i - y_i)(x_j - y_j) \right\}, \quad \varepsilon = \mp 1. \tag{14}$$

By the orientation of the line XY we will distinguish $D(X, Y) = -D(Y, X)$. Therefore, from (13) we have

$$M_Z = \lambda M_X + \xi M_Y - \varepsilon\lambda\xi D^2(X, Y). \tag{15}$$

Since X, Y and Z are collinear, we may write

$$D(X,Z) + D(Z,Y) = D(X,Y).$$

Thus, if we denote

$$\lambda = \frac{D(Z,Y)}{D(X,Y)}, \quad \xi = \frac{D(X,Z)}{D(X,Y)},$$

from (15) we get

$$M_Z = \frac{1}{D(X,Y)} \{D(Z,Y)M_X + D(X,Z)M_Y\} - \varepsilon D(X,Z)D(Z,Y). \tag{16}$$

The equivalent result for planar kinematics is given by Müller, [2].

Now, we consider that the points X and Y trace the same closed space curve. In this case, for the projection curves in the direction of $\mathbf{e}'$ we have $M_X = M_Y$. Then, from (16) we obtain

$$M_X - M_Z = \varepsilon D(X,Z)D(Z,Y). \tag{17}$$

Thus, we have Holditch-type result[1], [1], for the polar moments of inertia of projection curves. The equivalent result for projection areas under the 1-parameter closed spatial motions is given by [3]. So, we may give the following theorem.

THEOREM 2. Let us consider a line segment with the constant length. If the endpoints of the line segment trace the same space curve in R', then a different point on this segment traces another space curve. The difference between the polar moments of inertia of the projection curves (in the direction of unit vector $\mathbf{e}'$) of these space curves depends on the distances (in the sense of (14)) of the chosen point from the endpoints.

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[3] H. R. Müller, Erweiterung des Satzes von Holditch für geschlossene Raumkurven, Abh. Braunschw. Wiss. Ges., 31(1980), 129–135.

[1] The classical Holditch Theorem: If the endpoints X, Y of a segment of fixed length are rotated once on an oval, then a given point Z of this segment, with $\overline{XZ} = a$, $\overline{ZY} = b$, describes a closed, not necessarily convex, curve. The area of the ring-shaped domain bounded by the two curves is πab.