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EXISTENCE OF SOLUTIONS OF A NONLOCAL BOUNDARY VALUE PROBLEM*

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Abstract

In this paper, we discuss the existence of nontrivial and nonnegative solutions of a nonlocal boundary value problem for a one-dimensional p-Laplacian equation with nonlinear sources. The proof is based on a fixed point theorem.

1 Introduction

In the short announcement [1], two existence results are stated, but without proofs, regarding nontrivial and nonnegative solutions of the p-Laplacian equation

$$(\phi_p(u'))' + h(r)f(u,u') = 0, \quad r \in [0,1] \tag{1}$$

with the nonlocal boundary value condition

$$\phi_p(u'(1)) = \int_0^1 \phi_p(u'(s))dg(s) \tag{2}$$

and the natural boundary value condition

$$u(0) = 0. \tag{3}$$

where $\phi_p(s) = |s|^{p-2}s$, $p \geq 2$, $f : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$, $h : [0,1] \to (0,+\infty)$, $g : [0,1] \to [0,+\infty)$ and the integral in (2) is meant in the Riemann-Stieljes sense. In this paper, we intend to provide the full details leading to these two results.

Nonlocal boundary value problems of this form were first considered by Bitsadze [2], and later by Karakostas and Tsamatos [3], [4], Cao and Ma [5], Il'in and Moiseev [6], etc. Among those, Karakostas and Tsamatos [3] considered the following ordinary differential equation

$$x''(t) + q(t)f(x(t), x'(t)) = 0,$$

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which corresponds to the special case $p = 2$ of the equation (1), with the nonlocal boundary value conditions

$$x'(1) = \int_0^1 x'(s)dg(s), \quad x(0) = 0,$$

and proved the existence of the nonnegative solutions.

As in Karakostas and Tsamatos [3], we need some assumptions on the functions appeared in our problem. Assume that

(H1) $f : [0, +\infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $f(0, 0) > 0$ and

$$f(v, w) \geq 0, \quad \forall v, w > 0,$$

$g : [0, 1] \to \mathbb{R}$ is a continuous nondecreasing function with $0 = g(0) \leq g(1) < 1$, and

$$0 < h(r) \leq M, \quad \forall r \in [0, 1],$$

where M is a positive constant.

(H2) There exists a nondecreasing function $z : \mathbb{R}^+ \to \mathbb{R}^+$, such that

$$f(v, w) \leq z(w), \quad \forall v, w \in \mathbb{R}^+,$$

and moreover,

$$\liminf_{\tau \to +\infty} \int_{\phi_q\{[1/(1-g(1))]Mz(\tau)\|g\|_{L^1}\}}^{\tau} \frac{\phi_p'(r)}{z(r)} dr > M,$$

where $1/p + 1/q = 1$.

(H3)

$$\inf_{L>0} \frac{N(L)^{q-1}}{L} < \frac{1}{\sigma},$$

where

$$N(L) = \max\{f(u, v), u, v \in [0, L]\},$$

$$\sigma = \phi_q\left[\alpha \int_0^1 \int_s^1 h(\theta)d\theta dg(s) + \|h\|_{L^1}\right]$$

and $\alpha = 1/[1 - g(1)]$.

We will apply a fixed point theorem to obtain the following main results.

THEOREM 1. If (H1) and (H2) hold, then the problem (1)–(3) admits at least one nontrivial and nonnegative solution.

THEOREM 2. If (H1) and (H3) hold, then the problem (1)–(3) admits at least one nontrivial and nonnegative solution.

2 Proof of the main results

We are now in a position to prove the main results. A common technique to deal with this class of problems is based on fixed point theorems in cones and especially on the following well-known fixed-point theorem due to Krasnoselskii [3].

LEMMA 1. Let E be a Banach space and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let

$$A : K \cap (\Omega_2 \backslash \overline{\Omega}_1) \to K$$

be a completely continuous operator such that either

$$\|Au\| \leq \|u\|, \quad u \in K \cap \partial\Omega_1, \quad \text{and} \quad \|Au\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2,$$

or

$$\|Au\| \geq \|u\|, \quad u \in K \cap \partial\Omega_1, \quad \text{and} \quad \|Au\| \leq \|u\|, \quad u \in K \cap \partial\Omega_2.$$

Then the operator A has a fixed point in $K \cap (\Omega_2 \backslash \overline{\Omega}_1)$.

We want to use this fixed-point theorem to search for solutions of the problem (1)–(3). First, let us determine the corresponding notations. Set

$$E = \Big\{ u : [0,1] \to \mathbb{R} \,\Big|\, u' \text{ is continuous on } [0,1] \text{ and } u(0) = 0 \Big\}$$

and

$$K = \Big\{ u \in E \,\Big|\, u \geq 0, u \text{ is nondecreasing, } u' \text{ is no-increasing} \Big\}.$$

It is easy to see that E is a Banach space with the norm defined by

$$\|u\| = \sup\{|u'(r)| : r \in [0,1]\}.$$

It can be seen that if $u \in K$ and $\lambda \geq 0$, then $\lambda u \in K$, if $\pm u \in K$, then $u = 0$, i.e., so K is a cone.

Next, let us specify the appropriate operator A. Define

$$(Au)(r) = \int_0^r \phi_q \Big[\frac{1}{1-g(1)} \int_0^1 \int_s^1 h(\theta) Z(u)(\theta) d\theta dg(s) + \int_s^1 h(\theta) Z(u)(\theta) d\theta \Big] ds,$$

where $\phi_q = \phi_p^{-1}$, $1/p + 1/q = 1$, and

$$Z(u)(r) = f(u(r), u'(r)).$$

It is easy to see that A is a completely continuous operator. Noticing the hypothesis (H1), we can find that $(Au)(0) = 0$, $(Au)(r) \geq 0$, $(Au)(r)$ is nondecreasing. For $(Au)''(r) \leq 0$, we claim that $(Au)'(r)$ is no-increasing. That is A maps K into itself.

Let

$$\Phi(\psi) = \phi_q\Big[\frac{1}{1-g(1)}\int_0^1\int_s^1 h(\theta)Z(\int_0^{\cdot}\psi)(\theta)d\theta dg(s) \\ + \int_s^1 h(\theta)Z(\int_0^{\cdot}\psi)(\theta)d\theta\Big].$$

LEMMA 2. u is a solution of the problem (1)–(3) if and only if u is a fixed point of the operator A.

PROOF. Let u be a solution of the problem (1)–(3). Then the integration of the equation (1) implies that

$$\int_r^1 (\phi_p(u'(s)))'ds + \int_r^1 h(s)f(u(s),u'(s))ds = 0,$$

that is

$$\phi_p(u'(r)) = \phi_p(u'(1)) + \int_r^1 h(s)f(u(s),u'(s))ds. \tag{4}$$

Recalling (2), and using (4), we have

$$\begin{aligned}\phi_p(u'(1)) &= \int_0^1 \phi_p(u'(s))dg(s) \\ &= \int_0^1 \left[\phi_p(u'(1)) + \int_s^1 h(\theta)f(u(\theta),u'(\theta))d\theta\right] dg(s) \\ &= \phi_p(u'(1))g(1) + \int_0^1\int_s^1 h(\theta)f(u(\theta),u'(\theta))d\theta dg(s).\end{aligned}$$

So we obtain

$$\phi_p(u'(1)) = \alpha\int_0^1\int_s^1 h(\theta)f(u(\theta),u'(\theta))d\theta dg(s), \tag{5}$$

where $\alpha = \dfrac{1}{1-g(1)}$. Substituting (5) into (4), we obtain

$$\phi_p(u'(r)) = \alpha\int_0^1\int_s^1 h(\theta)f(u(\theta),u'(\theta))d\theta dg(s) + \int_r^1 h(\theta)f(u(\theta),u'(\theta))d\theta,$$

that is

$$u'(r) = \phi_q\Big(\alpha\int_0^1\int_s^1 h(\theta)f(u(\theta),u'(\theta))d\theta dg(s) + \int_r^1 h(\theta)f(u(\theta),u'(\theta))d\theta\Big).$$

Integrating it from 0 to r, and using the condition (3), we see that

$$u(r) = \int_0^r \phi_q\Big(\alpha\int_0^1\int_s^1 h(\theta)f(u(\theta),u'(\theta))d\theta dg(s) \\ + \int_s^1 h(\theta)f(u(\theta),u'(\theta))d\theta\Big)ds.$$

Recalling the definition of operator A, we see that the u is a fixed point of the operator A.

On the other hand, if u is a fixed point of operator A, that is

$$\begin{aligned} u(r) &= (Au)(r) \\ &= \int_0^r \phi_q\Big[\frac{1}{1-g(1)}\int_0^1\int_s^1 h(\theta)Z(u)(\theta)d\theta dg(s) \\ &+ \int_s^1 h(\theta)Z(u)(\theta)d\theta\Big]ds, \end{aligned}$$

where $Z(u)(r) = f(u(r), u'(r))$. Hence, we have

$$\begin{aligned} u(r) = \int_0^r \phi_q\Big(\alpha\int_0^1\int_s^1 h(\theta)f(u(\theta), u'(\theta))d\theta dg(s) \\ + \int_s^1 h(\theta)f(u(\theta), u'(\theta))d\theta\Big)ds, \end{aligned}$$

which implies that u is a solution of the problem (1)–(3).

LEMMA 3. If (H1) hold, there exists $m > 0$, such that for all $u \in K$ with $\|u\| = m$, we have $\|Au\| \geq \|u\|$.

PROOF. We argue by contradiction. For every positive integer n, there exists a function $u_n \in K$ with $\|u_n\| = 1/n$ and $\|Au_n\| < \|u_n\|$. Let $\psi_n = u_n'$. Then for all n and every $s \in [0, 1]$, we have

$$0 \leq \psi_n(s) \leq \psi_n(0) = \|u_n\| = \frac{1}{n},$$

which implies that $\psi_n \to 0$ in E. So we have

$$\begin{aligned} 0 &= \lim_{n\to\infty} \psi_n(0) \geq \lim_{n\to\infty} \Phi(\psi_n) = \Phi(0) \\ &= |f(0,0)|^{q-2}f(0,0)\phi_q\Big[\frac{1}{1-g(1)}\int_0^1\int_s^1 h(\theta)d\theta dg(s) + \|h(r)\|_{L^1}\Big], \end{aligned}$$

which is a contrary to (H1).

Now we define

$$f_n(v, w) = \min\{f(v, w), n^{\frac{1}{q-1}}\}.$$

Considering the problem

$$(\phi_p(u'))' + h(r)f_n(u, u') = 0, \quad r \in [0, 1], \tag{6}$$

with the conditions (2) and (3). And let A_n and Φ_n be the operators corresponding to A and Φ, we can get the following lemma.

LEMMA 4. For each n, if (H1) hold, then the problem (6), (2) and (3) has at least one nontrivial and nonnegative solution.

PROOF. For each n, by Lemma 2, there exists a positive real number m_n, such that $\forall u \in K$ with $\|u\| = m_n$, it holds $\|A_n u\| \geq \|u\|$. Moreover, if $u \in K$ satisfies

$$\|u\| = n\phi_q\Big[\frac{1}{1-g(1)}\int_0^1\int_s^1 h(\theta)d\theta dg(s) + \|h(r)\|_{L^1}\Big] \equiv C_n,$$

then it is easy to see that

$$\|A_n u\| = \Phi_n u' \leq C_n = \|u\|.$$

Hence by Lemma 2, there exists a nontrivial and nonnegative solution u_n of the problem (6), (2) and (3) and such that $m_n \leq \|u_n\| \leq C_n$.

LEMMA 5. If f, h, g satisfy the assumptions (H1) and (H3), then there exits a constant $T > 0$, such that for all $u \in K$ with $\|u\| = T$, we have $\|Au\| \leq \|u\|$.

PROOF. By (H3), there exits a constant $T > 0$ such that

$$\frac{N(T)^{q-1}}{T} \leq \frac{1}{\sigma}.$$

It is obviously that for all $u \in K$ with $\|u\| = u'(0) = T$, we have

$$0 \leq u(r), \quad u'(r) \leq T, \qquad \forall r \in [0,1].$$

Thus

$$\begin{aligned}\|Au\| &= (Au)'(0)\\ &= \phi_q\left[\alpha\int_0^1\int_s^1 h(\theta)f(u(r),u(r)')d\theta dg(s) + \int_0^1 h(\theta)f(u(r),u(r)')d\theta\right]\\ &\leq N(T)^{q-1}\phi_q\left[\alpha\int_0^1\int_s^1 h(\theta)d\theta dg(s) + \int_0^1 h(\theta)d\theta\right]\\ &= N(T)^{q-1}\sigma \leq T = \|u\|.\end{aligned}$$

Now we can give the proof of the main results.

PROOF OF THEOREM 1. We first prove that the set $\{u_n\}$ in Lemma 2 is a precompact subset of E. We want to utilize the classical Arzela-Ascoli Theorem. Thus, it is enough to show that the sets $\{u_n'\}$ and $\{u_n''\}$ are bounded. Also we should notice that for all n, $u_n(0) = 0$.

Let n be a fixed index and define

$$y_n = u_n'.$$

By the fact that $y_n \geq 0 \geq y_n'$, and for every $r \in [0,1]$, we have

$$0 \leq -y_n'(r) \leq \frac{h(r)z(y_n(r))}{\phi_p'(y_n(r))} \leq \frac{Mz(y_n(r))}{\phi_p'(y_n(r))}. \tag{7}$$

Then

$$-\frac{y_n'(r)\phi_p'(y_n(r))}{z(y_n(r))} \le M.$$

Integrating it from 0 to 1, then

$$-\int_0^1 \frac{y_n'(r)\phi_p'(y_n(r))}{z(y_n(r))}dr \le M.$$

That is

$$\int_{y_n(1)}^{y_n(0)} \frac{\phi_p'(t)}{z(t)}dt \le M. \tag{8}$$

On the other hand, by (2) and $g(0) = 0$, we have

$$\begin{aligned}
\phi_p(y_n(1)) &= \int_0^1 \phi_p(y_n(s))dg(s) \\
&= \phi_p(y_n(s))g(s)\Big|_0^1 - \int_0^1 g(s)d(\phi_p(y_n(s))) \\
&= \phi_p(y_n(1))g(1) + \int_0^1 g(s)h(s)f_n(u_n, u_n')ds, \\
&\le \phi_p(y_n(1))g(1) + M\int_0^1 g(s)z(y_n(s))ds.
\end{aligned}$$

Then

$$\phi_p(y_n(1)) \le \alpha M z(y_n(0))\|g\|_{L^1},$$

that is

$$y_n(1) \le \phi_q\Big[\alpha M z(y_n(0))\|g\|_{L^1}\Big].$$

Recalling (8), we have

$$\int_{\phi_q[\alpha M z(y_n(0))\|g\|_{L^1}]}^{y_n(0)} \frac{\phi_p'(t)}{z(t)}dt \le M.$$

Now, if $\{y_n(0)\}$ is not bounded, by taking a subsequence, if necessary, we can assume that $y_n(0) \to +\infty$. This fact implies that

$$\liminf_{\tau\to+\infty} \int_{\phi_q\{[1/(1-g(1))]Mz(\tau)\|g\|_{L^1}\}}^{\tau} \frac{\phi_p'(r)}{z(r)}dr \le M,$$

which contrary to (H2). Thus the sequence $\{y_n(0)\}$ is bounded and by (7), also the sequence $\{y_n'\}$ is bounded. Consequently, we can assume that $\{u_n\} \to u$ in E. This is equivalent to saying that $u_n \to u$ and $u_n' \to u'$ uniformly on $[0,1]$. Then from the equation (6), and by using continuous dependence arguments, we can easily obtain that u is a nontrivial and nonnegative solution of the problem (1)–(3). The proof is complete.

PROOF OF THEOREM 2. Let $\Omega_1 \triangleq \{u \in E : \|u\| < r_1\}$, $\Omega_2 \triangleq \{u \in E : \|u\| < r_2\}$, where $r_1 = \min\{m, T\}$, $r_2 = \max\{m, T\}$. By Lemma 3, Lemma 5 and Lemma 1, we know that A has a fixed point in $K \cap (\Omega_2 \backslash \overline{\Omega}_1)$. Then the theorem holds by Lemma 2.

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