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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF A SYSTEM OF VISCOUS CONSERVATION LAWS *

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Abstract

In this note, we study a special system of three parabolic equations, with initial condition and derive the asymptotic behaviour of the solution as time tends to infinity. This work is a generalization of a result of Hopf (1950) for the Burgers equation.

1 Introduction

In this paper we consider a system of partial differential equations of the form

$$u_t + \left(\frac{u^2}{2}\right)_x = \frac{\epsilon}{2}u_{xx}, v_t + (uv)_x = \frac{\epsilon}{2}v_{xx}, w_t + \left(\frac{v^2}{2} + uw\right)_x = \frac{\epsilon}{2}w_{xx}, \tag{1}$$

in $-\infty < x < \infty$, $t > 0$, supplemented with initial conditions at $t = 0$:

$$u(x, 0) = u_0(x), v(x, 0) = v_0(x), w(x, 0) = w_0(x), \tag{2}$$

where $u_0(x), v_0(x), w_0(x)$ are integrable functions and study the asymptotic behavior of its solution as t goes to infinity. When $v = 0, w = 0$, (1) is reduced to the Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = \frac{\epsilon}{2}u_{xx}. \tag{3}$$

Using Hopf-Cole transformation, Hopf in [2] linearized the equation (3) to the heat equation and explicit solution of the initial value problem was found. Also he constructed explicit entropy weak solution to the inviscid Burgers equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0,$$

with initial data, by passing the viscosity parameter ϵ to 0 in the explicit formula. Then he studied the asymptotic behavior of the solution of (3), with fixed $\epsilon > 0$, as t tends to infinity.

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The purpose of this note is to extend Hopf's analysis to the solution of (1) with initial condition (2). Indeed the system (1) is the vanishing viscosity approximation of the inviscid system of conservation laws,

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, v_t + (uv)_x = 0, w_t + \left(\frac{v^2}{2} + uw\right)_x = 0. \tag{4}$$

The system (4) is not strictly hyperbolic because the eigenvalues of the Jacobian matrix of the flux function $F(u,v,w) = (\frac{u^2}{2}, uv, \frac{v^2}{2} + uw)$ are all equal to u and so the classical theory of conservation laws of Lax [5] does not apply here. Even for initial data of Riemann type, (4) cannot be solved in the class of classical simple waves. When $v = 0$ in (4), the system reduces to

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, w_t + (uw)_x = 0. \tag{5}$$

Equations of the type (5) appear in the study of pressureless gas and was studied by LeFloch [6] and Joseph [3] and many others and constructed solutions in the class of Borel measures. But for the system (4), solutions cannot be found even in the class of Borel measures. In the paper [4], Joseph constructed solutions for (4), with general initial conditions, in a more singular class of function space, namely, in the class of generalized functions of Colombeau [1]. The regularization required in the construction of global solution was achieved by adding viscous terms to (4) and we arrived at the system (1). As in Hopf [2], in this note, we study another aspect of solution of (1) and (2), namely the large time behavior of the solution with fixed viscosity parameter $\epsilon > 0$. For notational convenience we drop the explicit dependence of ϵ on the solutions.

2 Explicit Formula and the Large Time Behavior of the Solution

It was shown in [4], that the equation (1) can be linearized using a generalized form of Hopf-Cole transformation. More precisely, we showed that

$$u = -\epsilon(\log(a))_x, v = -\epsilon\left(\frac{b}{a}\right)_x, w = \epsilon\left(-\frac{c}{a} + \frac{b^2}{2a^2}\right)_x, \tag{6}$$

is the solution of (1) with initial conditions (2) if a, b and c are solutions of the heat equation

$$a_t = \frac{\epsilon}{2}a_{xx}, b_t = \frac{\epsilon}{2}b_{xx}, c_t = \frac{\epsilon}{2}c_{xx} \tag{7}$$

with initial conditions

$$a(x,0) = e^{-\frac{U_0(x)}{\epsilon}}, b(x,0) = -\frac{V_0(x)}{\epsilon}e^{-\frac{U_0(x)}{\epsilon}}, c(x,0) = \left(\frac{V_0(x)^2}{2\epsilon^2} - \frac{W_0(x)}{\epsilon}\right)e^{-\frac{U_0(x)}{\epsilon}} \tag{8}$$

respectively, where

$$U_0(x)=\int_0^x u_0(y)dy, V_0(x)=\int_0^x v_0(y)dy, W_0(x)=\int_0^x w_0(y)dy. \tag{9}$$

Solving the heat equation (7) with initial conditions (8) we get a, b and c in the form

$$a(x,t)=\frac{1}{\sqrt{2\pi t\epsilon}}\int_{-\infty}^{+\infty} e^{-\frac{1}{\epsilon}[U_0(y)+\frac{(x-y)^2}{2t}]}dy, \tag{10}$$

$$b(x,t)=-\frac{1}{\epsilon\sqrt{2\pi t\epsilon}}\int_{-\infty}^{+\infty} V_0(y)e^{-\frac{1}{\epsilon}[U_0(y)+\frac{(x-y)^2}{2t}]}dy, \tag{11}$$

$$c(x,t)=\frac{1}{\sqrt{2\pi t\epsilon}}\int_{-\infty}^{+\infty}\left[\frac{V_0(y)^2}{2\epsilon^2}-\frac{W_0(y)}{\epsilon}\right] e^{-\frac{1}{\epsilon}[U_0(y)+\frac{(x-y)^2}{2t}]}dy. \tag{12}$$

We introduce the variable $\xi = x/\sqrt{t\epsilon}$ and the functions which appear in the asymptotic form of the solution, namely

$$A(\xi)=e^{-\frac{U_0(\infty)}{\epsilon}}\int_{-\infty}^{\xi} e^{-y^2/2}dy + e^{-\frac{U_0(-\infty)}{\epsilon}}\int_{\xi}^{\infty} e^{-y^2/2}dy. \tag{13}$$

$$B(\xi)=-\frac{V_0(\infty)}{\epsilon}e^{\frac{-U_0(\infty)}{\epsilon}}\int_{-\infty}^{\xi} e^{-y^2/2}dy - \frac{V_0(-\infty)}{\epsilon}e^{\frac{-U_0(-\infty)}{\epsilon}}\int_{\xi}^{\infty} e^{-y^2/2}dy. \tag{14}$$

$$\begin{aligned} C(\xi) \quad = \quad & \left(\frac{V_0^2(\infty)}{2\epsilon^2}-\frac{W_0(\infty)}{\epsilon}\right)e^{\frac{-U_0(\infty)}{\epsilon}}\int_{-\infty}^{\xi} e^{-y^2/2}dy \\ & + \left(\frac{V_0^2(-\infty)}{2\epsilon^2}-\frac{W_0(-\infty)}{\epsilon}\right)e^{\frac{-U_0(-\infty)}{\epsilon}}\int_{\xi}^{\infty} e^{-y^2/2}dy. \end{aligned} \tag{15}$$

With these notations, we shall prove the following theorem.

THEOREM 1. Assume that the initial data $u_0(x), v_0(x), w_0(x)$ are integrable, then the solution $(u(x,t), v(x,t), w(x,t))$ of (1) and (2) has the following asymptotic behavior as t tends to infinity:

$$\lim_{t\to\infty}\sqrt{t/\epsilon}u(x,t)=-\frac{A'(\xi)}{A(\xi)}, \tag{16}$$

$$\lim_{t\to\infty}\sqrt{t/\epsilon}v(x,t)=-\frac{B'(\xi)}{A(\xi)}+\frac{B(\xi)}{A(\xi)}\frac{A'(\xi)}{A(\xi)}, \tag{17}$$

$$\lim_{t\to\infty}\sqrt{t/\epsilon}w(x,t)=-\frac{C'(\xi)}{A(\xi)}+\frac{C(\xi)}{A(\xi)}\frac{A'(\xi)}{A(\xi)}+\frac{B(\xi)}{A(\xi)}\frac{B'(\xi)}{A(\xi)}-\frac{B(\xi)}{A(\xi)}\frac{B(\xi)}{A(\xi)}\frac{A'(\xi)}{A(\xi)}, \tag{18}$$

uniformly with respect to the variable $\xi = \frac{x}{\sqrt{t\epsilon}}$.

PROOF. To prove this theorem, we write the formula (6) for the solution in a convenient way:

$$u(x,t) = -\epsilon\frac{a_x}{a}, \tag{19}$$

$$v(x,t) = -\epsilon\frac{b_x}{a} + \epsilon\frac{b}{a}\frac{a_x}{a}, \tag{20}$$

$$w(x,t) = -\epsilon\frac{c_x}{a} + \epsilon\frac{c}{a}\frac{a_x}{a} + \epsilon\frac{b}{a}\frac{b_x}{a} - \epsilon\frac{b}{a}\frac{b}{a}\frac{a_x}{a}. \tag{21}$$

Note that typical terms in these expressions are of the form

$$\phi(x,t) = \frac{1}{\sqrt{2\pi t\epsilon}}\int_{-\infty}^{\infty}\phi_0(y)e^{-\frac{1}{\epsilon}[U_0(y)+\frac{(x-y)^2}{2t}]}dy \tag{22}$$

and its derivatives, where ϕ_0 has a finite limit as x tends to $+\infty$ and $-\infty$. So we first study the asymptotic behavior of $\phi(x,t)$ and $\phi_x(x,t)$.

We study the asymptotic behavior of these functions keeping the variable $\xi = x/\sqrt{\epsilon t}$ fixed. We make a change of variable $z = \frac{\sqrt{\epsilon t}\xi - y}{\sqrt{\epsilon t}}$ and renaming z as y, we get

$$\phi(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_0(\sqrt{\epsilon t}(\xi - y)e^{-[\frac{U_0(\sqrt{\epsilon t}(\xi-y)}{\epsilon}+y^2/2]}dy. \tag{23}$$

Now we split the integral in (23) in the following fashion

$$\begin{aligned}\sqrt{(2\pi)}\phi(x,t) \quad &= \quad \int_{-\infty}^{\xi-\delta}\phi_0(\sqrt{\epsilon t}(\xi - y))e^{-[\frac{1}{\epsilon}U_0(\sqrt{\epsilon t}(\xi-y)+y^2/2]}dy \\ &\quad + \int_{\xi+\delta}^{\infty}\phi_0(\sqrt{\epsilon t}(\xi - y))e^{-[\frac{1}{\epsilon}U_0(\sqrt{\epsilon t}(\xi-y)+y^2/2]}dy \\ &\quad + \int_{\xi-\delta}^{\xi+\delta}\phi_0(\sqrt{\epsilon t}(\xi - y))e^{-[\frac{1}{\epsilon}U_0(\sqrt{\epsilon t}(\xi-y)+y^2/2]}dy. \end{aligned} \tag{24}$$

Now we fix $\delta > 0$ and study each of these integrals as t tends to infinity, we get

$$\lim_{t\to\infty}\int_{-\infty}^{\xi-\delta}\phi_0(\sqrt{\epsilon t}(\xi - y)e^{-[\frac{1}{\epsilon}U_0(\sqrt{\epsilon t}(\xi-y)+y^2/2]}dy = e^{-\frac{U_0(\infty)}{\epsilon}}\phi_0(\infty)\int_{-\infty}^{\xi-\delta}e^{-y^2/2}dy,$$

$$\lim_{t\to\infty}\int_{\xi+\delta}^{\infty}\phi_0(\sqrt{\epsilon t}(\xi - y)e^{-[\frac{1}{\epsilon}U_0(\sqrt{\epsilon t}(\xi-y)+y^2/2]}dy = e^{-\frac{U_0(-\infty)}{\epsilon}}\phi_0(-\infty)\int_{\xi+\delta}^{\infty}e^{-y^2/2}dy,$$

$$\limsup_{t\to\infty}\left|\int_{\xi-\delta}^{\xi+\delta}\phi_0(\sqrt{\epsilon t}(\xi - y)e^{-\frac{1}{\epsilon}[U_0(\sqrt{\epsilon t}(\xi-y)+y^2/2]}dy\right| = O(\delta),$$

uniformly with respect to ξ. Now first let t tends to infinity and then δ tends to 0, in (24) we get

$$\lim_{t\to\infty}\sqrt{2\pi}\phi(x,t)=e^{\frac{-U_0(\infty)}{\epsilon}}\phi_0(\infty)\int_{-\infty}^{\xi}e^{-y^2/2}dy+e^{\frac{-U_0(-\infty)}{\epsilon}}\phi_0(-\infty)\int_{\xi}^{\infty}e^{-y^2/2}dy. \tag{25}$$

This limit is valid uniformly for $\xi\in R^1$ and for the x-derivative we have

$$\lim_{t\to\infty}\sqrt{2\pi\epsilon t}\phi_x(x,t)=(\phi(\infty)e^{\frac{-U_0(\infty)}{\epsilon}}-\phi(-\infty)e^{\frac{-U_0(-\infty)}{\epsilon}})e^{-\xi^2/2}. \tag{26}$$

Using (25) and (26) in (10)-(12), we get

$$\lim_{t\to\infty}\sqrt{2\pi}a(x,t)=e^{\frac{-U_0(\infty)}{\epsilon}}\int_{-\infty}^{\xi}e^{-y^2/2}dy+e^{\frac{-U_0(-\infty)}{\epsilon}}\int_{\xi}^{\infty}e^{-y^2/2}dy=A(\xi), \tag{27}$$

$$\lim_{t\to\infty}\sqrt{2\pi\epsilon t}a_x(x,t)=(e^{\frac{-U_0(\infty)}{\epsilon}}-e^{\frac{-U_0(-\infty)}{\epsilon}})e^{-\xi^2/2}=A'(\xi), \tag{28}$$

$$\begin{aligned}\lim_{t\to\infty}\sqrt{2\pi}b(x,t) &= -\frac{1}{\epsilon}e^{\frac{-U_0(\infty)}{\epsilon}}V_0(\infty)\int_{-\infty}^{\xi}e^{-y^2/2}dy\\ &\quad -\frac{1}{\epsilon}e^{\frac{-U_0(-\infty)}{\epsilon}}V_0(-\infty)\int_{\xi}^{\infty}e^{-y^2/2}dy\\ &= B(\xi),\end{aligned} \tag{29}$$

$$\lim_{t\to\infty}\sqrt{2\pi\epsilon t}.b_x(x,t)=-\frac{1}{\epsilon}\left(V_0(\infty)e^{\frac{-U_0(\infty)}{\epsilon}}-V_0(-\infty)e^{\frac{-U_0(-\infty)}{\epsilon}}\right)e^{-\xi^2/2}=B'(\xi), \tag{30}$$

$$\begin{aligned}\lim_{t\to\infty}\sqrt{2\pi}c(x,t) &= \left(\frac{V_0^2(\infty)}{2\epsilon^2}-\frac{W_0(\infty)}{\epsilon}\right)e^{\frac{-U_0(\infty)}{\epsilon}}\int_{-\infty}^{\xi}e^{-y^2/2}dy\\ &\quad +\left(\frac{V_0^2(-\infty)}{2\epsilon^2}-\frac{W_0(-\infty)}{\epsilon}\right)e^{\frac{-U_0(-\infty)}{\epsilon}}\int_{\xi}^{\infty}e^{-y^2/2}dy\\ &= C(\xi),\end{aligned} \tag{31}$$

$$\begin{aligned}\lim_{t\to\infty}\sqrt{2\pi\epsilon t}c_x(x,t) &= \left(\frac{V_0^2(\infty)}{2\epsilon^2}-\frac{W_0(\infty)}{\epsilon}\right)e^{-\frac{U_0(\infty)}{\epsilon}}e^{-\xi^2/2}\\ &\quad -\left(\frac{V_0^2(-\infty)}{2\epsilon^2}-\frac{W_0(-\infty)}{\epsilon}\right)e^{\frac{-U_0(-\infty)}{\epsilon}})e^{-\xi^2/2}\\ &= C'(\xi).\end{aligned} \tag{32}$$

These limits are valid uniformly for $\xi\in R^1$. We observe that $A(\xi)>0$ and hence letting t tends to infinity in (19)-(21) and using (27)-(32) we get the asymptotic form (16)-(18). The proof of the theorem is complete.

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