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OPTIMAL CONTROL OF A PRODUCTION SYSTEM WITH INVENTORY-LEVEL-DEPENDENT DEMAND *

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Abstract

Two production systems with inventory-level-dependent demand are considered and Pontryagin maximum principle is used to determine the optimal control.

1 Introduction

It has been observed that neglecting the effect of system parameters on inventory systems leads to a poor performance and unsatisfactory management. Thus, the class of systems where some dependence among the system parameters exists have received the attraction of many researchers. The dependence of the consumption rate on the on-hand inventory is, without doubt, the dependence that received the most attention and the literature on the subject is abundant. Among the most recent ones are [1, 9].

It has also been observed that the deterioration of items plays an important role in the inventory management. Thus, another class of systems was developed taking into account items deterioration. The literature on this subject is immense and an excellent survey, in which deteriorating inventory systems are thoroughly classified, has recently been done in [3]. Concerning cost parameters, the traditional approach in most models is to keep them constant. This assumption is somewhat unrealistic. Nonlinear holding costs have been introduced in [5] and were then considered in a few works such as [2].

All the models cited above are EOQ-type or extended EOQ-type models. EOQ-type models assume that inventory items are unaffected by time and replenishment is done instantaneously. Since this ideal situation is generally not applicable, extended (or generalized)-type models have been introduced to study dynamic inventory systems. These models assume that such system parameters as the demand rate, the production rate, or the deterioration rate vary with time; see the survey [3] for references on the subject. To cater for the dynamic behavior of production inventory systems, control theory has been successfully applied by some researchers; see for instance [4, 7, 8].

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In the present paper, we develop a first model in which the dynamic demand is a general functional of time and of the amount of on-hand stock. We have focused on the analysis of a production inventory system in which the nonlinear holding and production costs are treated as general functionals of the inventory level and production rate, respectively. We then extend this first model to an even more general model in which items deterioration is taken into account. The deterioration rate is also a general functional of time and of the amount of on-hand stock. For both models, we utilize optimal control theory to obtain an optimal control policy. The rest of the paper is organized as follows. Section 2 describes the first inventory model and develops the optimal control problem and its solution. A similar development is conducted in Section 3 for the second model. Section 4 concludes the paper.

2 Model Without Item Deterioration

Let us consider a manufacturing firm producing a single product. We assume that the decision horizon that the manager faces is finite, of length T. A finite planning horizon is interesting and appropriate because many firms are concerned with short and/or intermediate term market activities.

For $t \geq 0$, let $I(t)$ be the inventory level at time t, and let $D(t, I(t))$ and $h(I(t))$ be the corresponding demand rate and holding cost rate, respectively. Let $K(P(t))$ denote the cost rate corresponding to a production rate $P(t)$ at time t. Let $\rho \geq 0$ be the discount rate. All functions are assumed to be non-negative and continuous differentiable.

Given $T > 0$, the optimal control problem we are considering is:

$$(\mathcal{P}) \qquad \begin{cases} \displaystyle\min_{P(t)\geq D(t,I(t))} J(P,I) = \int_0^T e^{-\rho t}\Big\{h(I(t)) + K(P(t))\Big\}dt \\ \displaystyle\frac{d}{dt}I(t) = P(t) - D(t,I(t)), \quad I(0) = I_0, \quad I(T) = I_T. \end{cases}$$

The model is represented as an optimal control problem with one state variable (inventory status) and one control variable (rate of manufacturing). Since demand occurs at rate D and production occurs at the controllable rate P, it follows that $I(t)$ evolves according to the above dynamics (or state equation). The constraint $P(t) \geq D(t, I(t))$ with the state equation ensure $I(t) \geq I_0$ and I is nondecreasing. Therefore, shortages are not allowed in our study.

Using the Pontryagin maximum principle (see [6]), the necessary conditions for (P^*, I^*) to be an optimal solution of problem $(\mathcal{P})$ are that there should exist a constant β, a continuous and piecewise continuously differentiable function λ and a piecewise continuous function μ, called the adjoint and Lagrange multiplier functions, respectively, such that

$$H(t, I^*, P^*, \lambda) \geq H(t, I^*, P, \lambda), \text{ for all } P(t) \geq D(t, I^*(t)), \tag{1}$$

$$-\frac{d}{dt}\lambda(t) = \frac{\partial}{\partial I}L(t, I, P, \lambda, \mu), \tag{2}$$

$$I(0) = I_0, \qquad \lambda(T) = \beta, \tag{3}$$

$$\frac{\partial}{\partial P} L(t, I, P, \lambda, \mu) = 0, \tag{4}$$

$$P(t) - D(t, I(t)) \geq 0, \quad \mu(t) \geq 0, \quad \mu(t)\Big[P(t) - D(t, I(t))\Big] = 0, \tag{5}$$

where

$$H(t, I, P, \lambda) = -e^{-\rho t}\Big\{h(I(t)) + K(P(t))\Big\} + \lambda(t)\Big\{P(t) - D(t, I(t))\Big\}, \tag{6}$$

is the Hamiltonian function and

$$L(t, I, P, \lambda, \mu) = -e^{-\rho t}\Big\{h(I(t)) + K(P(t))\Big\} + \Big[\lambda(t) + \mu(t)\Big]\Big\{P(t) - D(t, I(t))\Big\}, \tag{7}$$

is the Lagrangian function. Equation (2) is equivalent to

$$\frac{d}{dt}\lambda(t) = e^{-\rho t}\frac{d}{dI}h(I(t)) + [\lambda(t) + \mu(t)]\,\frac{\partial}{\partial I}D(t, I(t)). \tag{8}$$

Equation (4) is equivalent to

$$\lambda(t) + \mu(t) = e^{-\rho t}\frac{d}{dP}K(P(t)). \tag{9}$$

Now, consider Equation (5). Then for any t, we have either $P(t) - D(t, I(t)) = 0$ or $P(t) - D(t, I(t)) > 0$.

<u>Case 1</u>: $P(t) - D(t, I(t)) = 0$ on some subset S of $[0, T]$. Then $\frac{d}{dt}I(t) = 0$ on S. In this case I^* is obviously constant on S and

$$P^*(t) = D(t, I^*(t)), \quad \text{for all } t \in S. \tag{10}$$

Substituting Equation (9) into (8) yields

$$\frac{d}{dt}\lambda(t) = e^{-\rho t}\left[\frac{d}{dI}h(I^*(t)) + \frac{d}{dP}K(P^*(t))\frac{\partial}{\partial I}D(t, I^*(t))\right].$$

Integrating this equation we get an explicit form of the adjoint function λ and of the constant β. Then an explicit form of Lagrange multiplier function μ can be obtained from Equation (9). Note that if the obtained function μ is not nonnegative, then the solutions given in Equation (10) are not acceptable.

<u>Case 2</u>: $P(t) - D(t, I(t)) > 0$ for $t \in [0, T]\backslash S$. Then $\mu(t) = 0$ on $[0, T]\backslash S$. In this case the necessary conditions (3), (8), and (9) become

$$\frac{d}{dt}\lambda(t) = e^{-\rho t}\frac{d}{dI}h(I(t)) + \lambda(t)\frac{\partial}{\partial I}D(t, I(t)), \quad I(0) = I_0, \quad \lambda(T) = \beta,$$

and

$$\lambda(t) = e^{-\rho t}\frac{d}{dP}K(P(t)).$$

Combining these equations with the state equation yields the following second order differential equation:

$$\frac{d}{dt}P(t)\frac{d^2}{dP^2}K(P) - \left[\rho + \frac{\partial}{\partial I}D(t,I)\right]\frac{d}{dP}K(P) = \frac{d}{dI}h(I), I(0) = I_0, \frac{d}{dP}K(P(T)) = \beta e^{\rho T} \qquad (11)$$

For illustration purposes, let us assume $K(P) = \frac{KP^2}{2}, h(I) = \frac{hI^2}{2}$, and $D(t,I) = d_1(t)+d_2I$, where K, h, and d_2 are positive constants. For these functions the necessary conditions for (P^*, I^*) to be an optimal solution of problem $(\mathcal{P})$ become

$$\frac{d^2}{dt^2}I(t) - \rho\frac{d}{dt}I(t) - \left[\frac{h}{K} + d_2(\rho + d_2)\right]I(t) = (\rho + d_2)d_1(t) - \frac{d}{dt}d_1(t),\ I(0) = I_0,\ I(T) = I_T. \quad (12)$$

This two-point boundary value problem $(\mathcal{P}_{TPBV})$ is solved in the next proposition.

PROPOSITION 1. The solution I^* of $(\mathcal{P}_{TPBV})$ is given by

$$I^*(t) = a_1e^{m_1t} + a_2e^{m_2t} + Q(t), \qquad (13)$$

and its corresponding P^* is given by

$$P^*(t) = a_1(m_1 + d_2)e^{m_1t} + a_2(m_2 + d_2)e^{m_2t} + \frac{d}{dt}Q(t) + d_2Q(t) + d_1(t) \qquad (14)$$

where the constants a_1, a_2, m_1, and m_2 are given in the proof below, and $Q(t)$ is a particular solution of Equation (12).

PROOF. We solve Equation (12) by the standard method. Its characteristic equation $m^2 - \rho m - \left[\frac{h}{K} + (\rho + d_2)d_2\right] = 0$, has two real roots of opposite signs, given by

$$m_1 = \frac{1}{2}\left(\rho - \sqrt{\rho^2 + 4\left[\frac{h}{K} + (\rho + d_2)d_2\right]}\right) < 0 \text{ and } m_2 = \frac{1}{2}\left(\rho + \sqrt{\rho^2 + 4\left[\frac{h}{K} + (\rho + d_2)d_2\right]}\right) > 0,$$

and therefore $I^*(t)$ is given by (13), where $Q(t)$ is a particular solution of (12). The initial and terminal conditions are used to determine the constants a_1 and a_2 as follows. From the initial and terminal conditions we have $a_1 + a_2 + Q(0) = I_0$ and $a_1e^{m_1T} + a_2e^{m_2T} + Q(T) = I_T$. By putting $b_1 = I_0 - Q(0)$ and $b_2 = I_T - Q(T)$, we obtain the following system of two linear equations in two unknowns

$$\begin{aligned} a_1 + a_2 &= b_1 \\ a_1e^{m_1T} + a_2e^{m_2T} &= b_2, \end{aligned}$$

which has the following unique solution

$$a_1 = \frac{b_2 - e^{m_2T}b_1}{e^{m_1T} - e^{m_2T}} \quad \text{and} \quad a_2 = \frac{b_1e^{m_1T} - b_2}{e^{m_1T} - e^{m_2T}}.$$

The expression of P^* is deduced using the expression of I^* and the state equation.

From the above analysis we have the following theorem characterizing the optimal solution of $(\mathcal{P})$.

THEOREM 1. The optimal solution (P^*, I^*) of $(\mathcal{P})$ has the form given in Equation (10) on S, and the form in Equations (13) – (14) on $[0,T]\backslash S$.

EXAMPLE 1. Consider a production system with the following characteristics: initial and terminal inventory levels $I(0) = 0, I(T) = 10$; unit costs and discount factor $h = 0.1, K = 5$, and $\rho = 0$, respectively. The planning horizon is $T = 5$, and the stock-dependent demand is such that $d_2 = 0.1, d_1(t) = \cos(t) + 1$. Variations of the optimal production rate and optimal stock level are displayed in Figure 1. The optimal cost was found to be $J = 139.5014$. Changing the shape of the demand function by taking $d_1(t) = e^{-t}$ and keeping all other parameters unchanged yielded the graphs represented in Figure 2. The objective function value changed to $J = 87.9876$.

3 Model With Item Deterioration

In this section, we assume that the product deteriorates while in stock. For $t \geq 0$, let $\theta(t, I(t))$ be the deterioration rate at the inventory level $I(t)$ at time t. Keeping the same notation as in the previous section, the optimal control problem becomes:

$$(\mathcal{P}_\theta) \quad \begin{cases} \min\limits_{P(t) \geq D(t,I(t))+\theta(t,I(t))} J(P,I) = \int_0^T e^{-\rho t} \{h(I(t)) + c\,[P(t) - D(t,I(t))] + K(P(t))\}\, dt \\ \dfrac{d}{dt} I(t) = P(t) - D(t,I(t)) - \theta(t,I(t)), \quad I(0) = I_0, \quad I(T) = I_T, \end{cases}$$

where $c > 0$ is the unit cost. The necessary conditions (1)-(4) remain the same with

$$H(t,I,P,\lambda) = -e^{-\rho t}\Big[h(I) + c\,[P(t) - D(t,I)] + K(P)\Big] + \lambda(t)\Big[P(t) - D(t,I) - \theta(t,I)\Big], \tag{15}$$

$$L(t,I,P,\lambda,\mu) = H(t,I,P,\lambda(t)) + \mu(t)\Big[P(t) - D(t,I) - \theta(t,I)\Big], \tag{16}$$

while Equation (5) becomes

$$P(t) - D(t,I) - \theta(t,I) \geq 0, \quad \mu(t) \geq 0, \quad \mu(t)\Big[P(t) - D(t,I) - \theta(t,I)\Big] = 0, \tag{17}$$

Equations (2), (4), and (16) yield

$$\frac{d}{dt}\lambda(t) = e^{-\rho t}\left[\frac{d}{dI}h(I) - c\frac{d}{dI}D(t,I)\right] + [\lambda(t) + \mu(t)]\left[\frac{\partial}{\partial I}D(t,I) + \frac{\partial}{\partial I}\theta(t,I)\right]. \tag{18}$$

$$\lambda(t) + \mu(t) = e^{-\rho t}\left[\frac{d}{dP}K(P) + c\right]. \tag{19}$$

Now, consider Equation (17). Then, on some subset S of $[0,T]$, we have $P(t) - D(t,I(t)) - \theta(t,I(t)) = 0$ and the optimal control in this case is given by

$$P^*(t) = D(t,I^*(t)) + \theta(t,I^*(t)), \quad \text{for all } t \in S. \tag{20}$$

On the set $[0,T]\backslash S$, we have $P(t) - D(t,I(t)) - \theta(t,I(t)) > 0$. Using the same argument as in the previous section, we obtain the following second order differential equation:

$$\frac{d}{dt}P(t)\frac{d^2}{dP^2}K(P)-\left[\rho+\frac{\partial}{\partial I}D(t,I)+\frac{\partial}{\partial I}\theta(t,I)\right]\left[\frac{d}{dP}K(P)+c\right]=\frac{d}{dI}h(I)-c\frac{\partial}{\partial I}D(t,I), \quad (21)$$

and $I(0) = I_0, \frac{d}{dP}K(P(T)) = \beta e^{\rho T}$. Let us assume now $K(P) = \frac{KP^2}{2}$, $h(I) = \frac{hI^2}{2}$, $D(t,I) = d_1(t) + d_2 I$, and $\theta(t,I) = \theta_1(t) + \theta_2 I$ where K, h, d_2, and θ_2 are positive constants. Then the previous differential equation in P becomes the following second order differential equation in I

$$\frac{d^2}{dt^2}I(t)-\rho\frac{d}{dt}I(t)-\left[\frac{h}{K}+(d_2+\theta_2)(\rho+d_2+\theta_2)\right]I(t)=\alpha(t), \quad (22)$$

with $\alpha(t) = (\rho+d_2+\theta_2)(d_1(t)+\theta_1(t)) - \frac{d}{dt}d_1(t) - \frac{d}{dt}\theta_1(t) - cd_2$, and $I(0) = I_0, I(T) = I_T$. The solution of this two-point boundary value problem is given by Equation (13) with

$$\begin{aligned}
m_1 &= \frac{1}{2}\left[\rho-\sqrt{\rho^2+4\left[\frac{h}{K}+(\rho+d_2+\theta_2)(d_2+\theta_2)\right]}\right],\\
m_2 &= \frac{1}{2}\left[\rho+\sqrt{\rho^2+4\left[\frac{h}{K}+(\rho+d_2+\theta_2)(d_2+\theta_2)\right]}\right],\\
a_1 &= \frac{I_T-Q(T)-(I_0-Q(0))e^{m_2T}}{e^{m_1T}-e^{m_2T}},\\
a_2 &= \frac{(I_0-Q(0))e^{m_1T}-I_T+Q(T)}{e^{m_1T}-e^{m_2T}},
\end{aligned}$$

where $Q(t)$ is a particular solution of (22). The expression of P^* is deduced using I^* along with the state equation. Finally, as in Theorem 2.1, the optimal solution (P^*, I^*) of $(\mathcal{P}_\theta)$ is given on $[0,T]\backslash S$ by the solution of the differential equation (22) and its corresponding optimal production while on S, it has the form given in (20).

EXAMPLE 2. Consider the production system of Example 2.1 and let $I(T) = 20$ and the unit cost $c = 0.1$. The deterioration rate is such that $\theta_1(t) = \sin(t)+1, \theta_2 = 0.1$. The optimal control and state are displayed in Figure 3. The optimal objective function value is $J = 773.2404$. To assess the effect of the deterioration rate on the value of the optimal objective function, we set $\theta_1 = 0$ and varied the value of θ_2 from 0.0005 to 0.256. As shown by the table below, the resulting optimal cost increases as θ_2 increases.

θ_2	0.0005	0.001	0.002	0.004	0.008	0.016	0.032	0.064	0.128	0.256
J	426.97	449.14	450.54	453.37	459.05	470.53	493.98	542.78	647.62	883.77

4 Conclusion

Explicit optimal controls are obtained for two general inventory-level-dependent demand production models. These models can be extended in various ways. For example, instead of minimizing the total cost, one may want to maximize the total profit where the unit revenue rate is both function of time and of the inventory level.

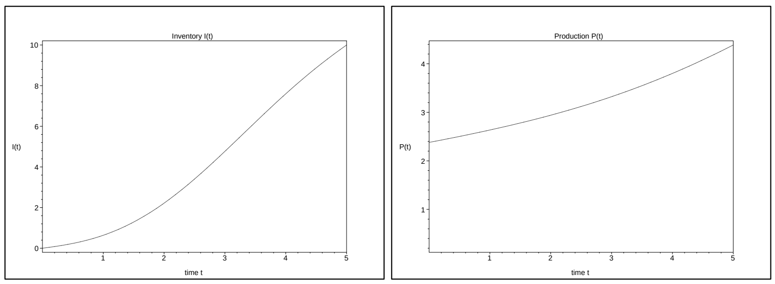

Figure 1. Figure 1. Variations of (P^*, I^*) as function of time t for $d_1(t) = \cos(t) + 1$.

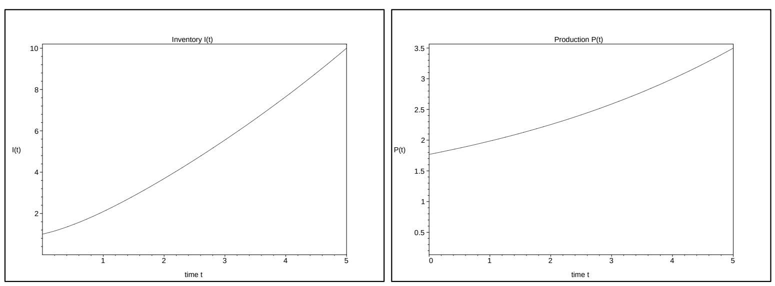

Figure 2. Variations of (P^*, I^*) for $d_1(t) = e^{-t}$.

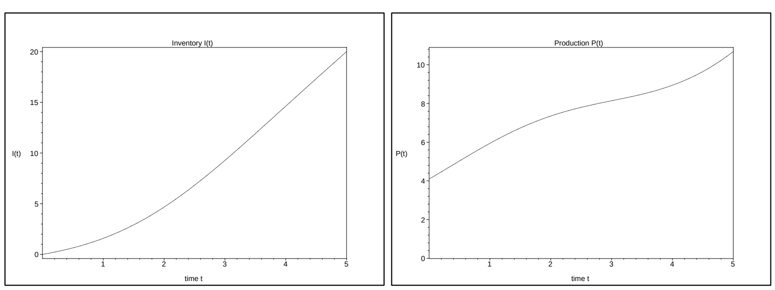

Figure 3. Variations of (P^*, I^*) as function of time t.

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