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SOME REMARKS ON SUBDIFFERENTIABILITY OF CONVEX FUNCTIONS*

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Abstract

In this paper, we study the subdifferentiability of convex functions with semi-closed epigraphs. This broad class includes convex proper lower semicontinuous functions, cs-convex functions and also cs-closed functions. Also, we show that a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$, defined on a Fréchet space and supposed only to be lower semicontinuous at $\bar{x} \in$ dom f is subdifferentiable at $\bar{x}$ under the Attouch-Brézis condition. The proof of these results is based on Baire's theorem.

1 Introduction

Let X be Hausdorff topological vector space and X^* its dual space. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Finding sufficient conditions ensuring that

$$\partial f(\bar{x}) \neq \emptyset, \tag{1}$$

for $\bar{x} \in$ dom f, is of crucial importance in convex analysis, optimization, mechanics, game theory and mathematical economics. Among such conditions let us mention the Attouch-Brézis condition [1] which assumes that the underlying space X is a Banach space, $f : X \to \mathbb{R}\cup\{+\infty\}$ is convex, proper and lower semicontinuous and $\mathbb{R}_+[\text{dom f}-\bar{x}]$ is a closed vector subspace. This condition has been weakened later in some sense by S. Simons [11], C. Zalinescu [12], [13] and C. Amara & M. Ciligot-Travain [2]. Simons via his open mapping stated (1) in the setting of metrizable locally convex real vector spaces by supposing f is cs-convex (rather than convex and lower semicontinuous) and $\mathbb{R}_+[\text{dom f} - \bar{x}]$ is a barreled linear vector subspace. Zalinescu proved (1) in the setting of Fréchet spaces under the assumption that f is cs-closed (rather than cs-convex) and $\mathbb{R}_+[\text{dom f} - \bar{x}]$ is a closed vector subspace. Amara & Ciligot-Travain in their recent paper [2] established (1) in the setting of locally convex linear spaces by supposing f is lower cs-closed (rather than cs-closed) and $\mathbb{R}_+[\text{dom f} - \bar{x}]$ is a metrizable barreled space. Let us note that a cs-convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ is cs-closed but the converse is true if its conjugate function f^* is assumed to be proper (see [12]).

The purpose of this note is to attempt to prove that statement (1) holds for a broad class of convex functions whose epigraphs are semi-closed (i.e. epigraph and its

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closure have the same topological interior) under the assumptions that $\mathbb{R}_+[\text{dom f} - \bar{x}]$ is a closed vector subspace and X is a Fréchet space. This broad class of convex functions includes convex lower semicontinuous functions, cs-convex functions and cs-closed functions. One may ask a natural question if a lower cs-closed function (see below the definition) is semi-closed? The answer seems to be unknown.

As mentioned above, the subdifferentiability of a convex function $f : X \to \mathbb{R}\cup\{+\infty\}$ at $\bar{x} \in$ dom f under the Attouch-Brézis condition requires that f is lower semicontinuous on the whole space X. Our goal is to attempt to weaken this requirement by supposing only that f is lower semicontinuous at $\bar{x}$. The main tool under which are based these results is Baire's theorem.

2 Preliminaries and Notations

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. In what follows, we denote by

$$\text{dom f} := \{x \in X : f(x) < +\infty\}$$

its effective domain, by

$$\text{Epi f} := \{(x, r) \in X \times \mathbb{R} : f(x) \leq r \}$$

its epigraph and by

$$[f \leq r] := \{x \in X : f(x) \leq r \}$$

its sublevel set at height r. The subdifferential of f at a point $\bar{x}$ is by definition

$$\partial f(\bar{x}) := \{x^* \in X^* : f(x) \geq f(\bar{x}) + \langle x^*, x - \bar{x}\rangle, \qquad \forall x \in X\}$$

where the symbol $\langle \cdot, \cdot \rangle$ stands for the duality between X and X^*. Let K be a subset of X, the cone that it generates is

$$\mathbb{R}_+K := \bigcup_{\lambda \geq 0} \lambda K.$$

Following [9], we say that K is cs-closed if whenever $(x_n)_{n\in\mathbb{N}}$ is a sequence in K and $(\alpha_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}^+$ with $\sum_{n=0}^{\infty} \alpha_n = 1$ and $x = \sum_{n=0}^{\infty} \alpha_n x_n$ exists in X, then $x \in K$. It is easy to see that every cs-closed subset is convex. A subset K is said to be semi-closed if K and its closure $\overline{K}$ have the same interior. Also, a subset K of a locally convex space X is said to be lower cs-closed if there exist a Fréchet space Y and a cs-closed subset A of $X \times Y$ such that $K = A_X$ where A_X denotes the projection of A on the space X. The following examples show that there are plenty of sets that are cs-closed or lower cs-closed or semi-closed (see [4] [6], [8], [9],[11],[12], [2]).

1. Any open convex subset is cs-closed.
2. In the space of all bounded real sequences, let P the set of sequences in which the first non-zero term is positive, together with zero. Then P is cs-closed.

3. A linear subspace is cs-closed if and only if it is sequentially closed, (in a metrizable space, if and only if it is closed).

4. Any convex closed subset is cs-closed.

5. In a metrizable space, every cs-closed subset is semi-closed.

6. In the case when X is a metrizable space, le us consider a linear subspace assumed to be neither closed nor dense in X. Hence it follows that L is not cs-closed but semi-closed (since int L =int $\bar{L} = \emptyset$).

7. Any convex subset with nonempty interior is semi-closed.

8. Let X be a Banach space, Y be a normed vector space and C be a closed convex subset of $X \times Y$. If the projection of C on X is bounded then the projection C_Y of C on Y is semi closed. This example constitutes in fact a fundamental tool for establishing the well known openness theorem due to S. Robinson [10] in Banach space.

9. The sum of two closed linear spaces is always lower cs-closed but may fail to be cs-closed.

Now, following [11] and [12] we set

DEFINITION 2.1. Let $f : X \to \mathbb{R} \cup \{+\infty\}$

1. We say that f is semi-closed if it is proper and its epigraph is semi-closed.

2. We say that f is cs-closed (resp. lower cs-closed) if it is proper and its epigraph is cs-closed (resp. lower cs-closed).

3. We say that f is cs-convex if f is proper and

$$f(x) \leq \liminf_{m \to +\infty} \sum_{n=0}^{m} \lambda_n f(x_n)$$

whenever, $\forall n \in \mathbb{N},\ \lambda_n \geq 0,\ x_n \in X,\ \sum_{n=0}^{\infty} \lambda_n = 1$ and $\sum_{n=0}^{\infty} \lambda_n x_n$ is convergent to x in X.

REMARK 2.1. Let us note that if $f : X \to \mathbb{R} \cup \{+\infty\}$ is proper, convex and lower semicontinuous then it is cs-convex. If f is cs-convex, then it is convex and Epi f is cs-closed. Conversely, C. Zalinescu in [12], has proved that when f^* is proper and f is cs-closed then f is cs-convex. The indicator function $\delta_C : X \to \mathbb{R} \cup \{+\infty\}$ (i.e. $\delta_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise) of every convex semi-closed subset of X (resp. of every cs-closed or lower cs-closed) is semi-closed (resp. is cs-closed or lower cs-closed).

PROPOSITION 2.1. In any topological vector space X we have: $f : X \to \mathbb{R} \cup \{+\infty\}$ is semi-closed if and only if its level sets $[f \leq \lambda]$ are semi-closed for any $\lambda \in \mathbb{R}$.

PROOF. ($\Longrightarrow$) It is obvious that $\text{int}([f \leq \lambda] \subset \text{int}(\overline{[f \leq \lambda]})$ for any $\lambda \in \mathbb{R}$. Conversely, let us take any $x \in \text{int}(\overline{[f \leq \lambda]})$, there exists some open neighbourhood V_x of x such that $V_x \subset \overline{[f \leq \lambda]}$. For any $y \in V_x$ we may choose a neighbourhood V_y of y such that $V_y \subset V_x$. By fixing any $z \in V_y$ we have $[f \leq \lambda] \cap W \neq \emptyset$ for any neighbourhood W of z and therefore we obtain

$$\text{Epi f} \cap W \times]\gamma - \epsilon, \gamma + \epsilon[\neq \emptyset, \quad \forall \epsilon > 0\ , \forall \gamma \in]\lambda + \epsilon, \lambda + 3\epsilon[$$

i.e., $(z, \gamma) \in \overline{\text{Epi f}}$ for any $z \in V_y$ and $\gamma \in (r_\epsilon - \epsilon, r_\epsilon + \epsilon)$ with $r_\epsilon := \lambda + 2\epsilon$. Hence $(y, r_\epsilon) \in \text{int}(\overline{\text{Epi f}})$. As Epi f is semi-closed, it follows that $(y, r_\epsilon) \in \text{int}(\text{Epi f})$ and hence we get $f(y) \leq r_\epsilon$ for any $y \in V_y$. By letting $\epsilon \longrightarrow 0$, we get $f(y) \leq \lambda,\ \forall y \in V_y$ i.e. $x \in \text{int}[f \leq \lambda]$ and therefore $[f \leq \lambda]$ is semi-closed for any $\lambda \in\ R$.

($\Longleftarrow$) In the same way as above, we will show that only $\text{int}(\overline{\text{Epi f}}) \subset \text{int}(\text{Epi f})$. For this, let $(x, \lambda) \in \text{int}(\overline{\text{Epi f}})$ i.e. there is some open neighbourhood V_x of x and $\alpha > 0$ such that $V_x \times (\lambda - \alpha, \lambda + \alpha) \subset \overline{\text{Epi f}}$. For any $y \in V_x$ there is some neighbourhood V_y of y such that $V_y \subset V_x$. By taking any $z \in V_y$ and any $\gamma \in (\lambda - \alpha, \lambda + \alpha)$ we have for any neighbourhood W of z and any $\epsilon > 0$

$$\text{Epi f} \cap W \times (\gamma - \epsilon, \gamma + \epsilon) \neq \emptyset$$

which implies $[f \leq \gamma + \epsilon] \cap W \neq \emptyset$, i.e., $z \in \overline{[f \leq \gamma + \epsilon]},\ \forall z \in V_y$ and hence $y \in \text{int}\overline{[f \leq \gamma + \epsilon]}$. As $[f \leq \gamma + \epsilon]$ is semi-closed, it follows that $y \in \text{int}[f \leq \gamma + \epsilon]$ for any $(y, \epsilon) \in V_x \times (0, +\infty)$ which yields

$$(y, \gamma) \in \text{Epi f}, \quad \forall (y, \gamma) \in V_x \times (\lambda - \alpha, \lambda + \alpha)$$

i.e., $(x, \lambda) \in \text{int}(\text{Epi f})$ and thus Epi f is semi-closed. The proof is complete.

COROLLARY 2.1. In a metrizable topological vector space, we have 1°) every cs-closed function is semi-closed, and 2°) every lower semicontinuous, proper and convex function is semi-closed.

Indeed, 1°) holds since any cs-closed subset of a metrizable topological linear space is semi-closed. 2°) holds since every convex closed subset is cs-closed.

PROPOSITION 2.2. If f is cs-closed then its level sets $[f \leq \lambda]$ are cs-closed.

PROOF. Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $[f \leq \lambda]$ and $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^+$ with $\sum_{n=0}^{\infty} \alpha_n = 1$ and $x = \sum_{n=0}^{\infty} \alpha_n x_n$ exists in X. Since

$$(x, \lambda) = \big(\sum_{n=0}^{\infty} \alpha_n x_n, \sum_{n=0}^{\infty} \alpha_n \lambda\big) = \sum_{n=0}^{\infty} \alpha_n (x_n, \lambda)$$

with $(x_n, \lambda) \in$ Epi f and Epi f is cs-closed hence it follows that $(x, \lambda) \in$ Epi f, i.e., $x \in [f \leq r]$.

REMARK 2.2. It is natural to ask ourselves the following question: does the converse of Proposition 2.2 remain true? The answer is negative with the following counterexample. Just take $X = \mathbb{R}$ and $f(x) = x^3$. Obviously, f is not convex but its level sets given by $[f \leq \lambda] = (-\infty, \lambda^{\frac{1}{3}})$ are convex and closed subsets of $\mathbb{R}$ for any $\lambda \in \mathbb{R}$, hence cs-closed. A particular subclass of cs-closed functions for which the converse holds is the class of functions whose epigraph is ideally convex (A is ideally convex if the condition of cs-closed sets one asks that the sequence $(x_n)_n$ is bounded). Then Epi f is ideally convex if, and only if, $[f \leq \lambda]$ is ideally convex for any $\lambda \in \mathbb{R}$.

3 The main result

Before stating our main result we will need in the sequel the following result.

LEMMA 3.1. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex proper function. If we assume that $\mathbb{R}_+[\text{dom f}]$ is a vector subspace of X then we have

$$\mathbb{R}_+[\text{dom f}] = \bigcup_{n,m \in \mathbb{N}^*} m[f \leq n].$$

PROOF. The desired result is obtained simply by observing that

$$\text{dom f} = \bigcup_{n \geq 1} [f \leq n].$$

Now, we are ready to state our main result.

THEOREM 3.1. Let X be a Fréchet space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex semi-closed function. If we suppose that $\mathbb{R}_+[\text{dom f}] = X$ then, $\partial f(0) \neq \emptyset$.

PROOF. Let us note that zero is in the interior of dom f. By Lemma 3.1, Baire's theorem, Proposition 2.1 and the definition of a semi-closed set, there exist $m, n \in \mathbb{N}^*$ such that $0 \in \text{int}\big(\overline{m[f \leq n]}\big) = \text{int}\big(m[f \leq n]\big)$. Therefore, it follows that f is bounded above on a neighbourhood of zero and since f is finite at zero and convex we obtain from a classical convex analysis result (see [5]) that f is subdifferentiable at zero i.e. $\partial f(0) \neq \emptyset$.

COROLLARY 3.1. Let X be a Fréchet space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex semi-closed function (resp. be a cs-closed or convex, proper and lower semicontinuous function). If $\mathbb{R}_+[\text{dom f} - \bar{x}] = X$ then, $\partial f(\bar{x}) \neq \emptyset$.

Indeed, it suffices to apply the above Theorem to the function $x \to f(x + \bar{x})$.

REMARK 3.1. 1°) Also a natural and classical question is then, does the result of Theorem 3.1 remain true under the weakened condition: $\mathbb{R}_+[\text{dom f}]$ is a closed vector subspace? The answer is no with the present definition of a semi-closed set. Just take X an infinite dimensional Banach space, $f : X \to \mathbb{R}$ a noncontinuous linear functional, $Y := X \times \mathbb{R}$ and $g : Y \to \mathbb{R} \cup \{+\infty\}$ defined by $g(x, t) := +\infty$ if $t \neq 0$ and $g(x, t) := f(x)$. It is easy to see that g is convex, semi-closed, $\mathbb{R}_+[\text{dom f}] = X \times \{0\}$ is a closed linear subspace and g is nowhere subdifferentiable.

2°) It is more natural to say that A is semi-closed if A and it closure $\overline{A}$ have the same interior with respect to the affine hull of $\overline{A}$. With this definition the result in Theorem 3.1 remains valid.

3°) Note that for a convex set A of X one has $\mathbb{R}_+ A = X$ if, and only if, 0 is in the interior of A. So the condition "$\mathbb{R}_+[\text{dom f} - \bar{x}] = X$" is equivalent to "$x$ is the interior of dom f" (for f convex , which is the case throughout the paper), condition which is much older than the Attouch-Brézis condition.

It is obvious that if f is subdifferentiable at $\bar{x} \in \text{dom f}$ then f is lower semicontinuous at $\bar{x}$ for any topological vector space X. On the other hand, it is well known that if the convex function f is lower semicontinuous on the whole space and the space is Fréchet then f is subdifferentiable at any point of its algebraic interior. In [1], Attouch and Brézis proved in the setting of Fréchet spaces the subdifferentiability of a convex function $f : X \to \mathbb{R} \cup \{+\infty\}$ at $\bar{x} \in \mathrm{dom\ f}$ under the condition that $\mathbb{R}_+[\mathrm{dom\ f} - \bar{x}]$ is a closed vector subspace and f is lower semicontinuous on the entire space X. In what follows, we will prove that the same result holds under the Attouch-Brézis condition by supposing only that f is lower semicontinuous at $\bar{x}$.

THEOREM 3.2. Let X be a Fréchet space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex proper function such that $\mathbb{R}_+[\mathrm{dom\ f}]$ is a closed vector subspace. Then i) $\partial f(0) \neq \emptyset$ if, and only if, ii) f is l.s.c at zero.

PROOF. $i) \Longrightarrow ii)$ is obvious. Let us consider $\bar{f}$ the closure of the convex function f i.e. the greatest l.s.c function $\leq f$. Obviously $\bar{f}$ is convex since $\mathrm{Epi\ \bar{f}} = \overline{\mathrm{Epi\ f}}$ (see [5]). As $Z := \mathbb{R}_+[\mathrm{dom\ f}] = \mathbb{R}_+[\mathrm{dom\ \bar{f}}]$ is a closed vector subspace of X hence by applying the same way used in the proof of Theorem 3.1 we obtain $\bar{f}_0$ is subddifferentiable at zero where $\bar{f}_0$ denotes the restriction of $\bar{f}$ over Z. Taking $x_0^* \in \partial \bar{f}_0(0)$, any continuous linear functional x^* extending x_0^* to all X is easily seen to be in $\partial \bar{f}(0)$. Since $\bar{f}(x) \leq f(x)$ for any $x \in X$ and $\bar{f}(0) = f(0)$ it results that any $x^* \in \partial \bar{f}(0)$ is in $\partial f(0)$ and the proof is complete.

COROLLARY 3.2. Let X be a Fréchet space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a convex proper function such that $\mathbb{R}_+[\mathrm{dom\ f} - \bar{x}]$ is a closed vector subspace. Then $\partial f(\bar{x}) \neq \emptyset$ if, and only if, f is l.s.c at $\bar{x}$.

REMARK 3.2. It will appear in a forthcoming paper [7] a study of convex duality dealing with this broad class of convex functions with semi-closed epigraphs.

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