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ON SOME FAMILIES OF AQSI RANDOM VARIABLES AND RELATED STRONG LAW OF LARGE NUMBERS *

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Abstract

We prove the strong law of large numbers for a class of asymptotically quadrant sub-independent random variables. The obtained result generalize the SLLN for pairwise independent and pairwise negatively quadrant dependent random variables.

1 Introduction

Chandra and Ghosal (cf. [2], [3]) introduced the notion of asymptotically quadrant sub-independent (AQSI) random variables (r.v.'s) in the following way.

DEFINITION 1. A sequence $(X_n)_{n\in N}$ of r.v.'s is said to be AQSI if there exists a sequence $(q(m))_{m\in N}$ of nonnegative numbers such that $q(m)\to 0$, as $m\to\infty$, and for every $i\neq j$

$$P(X_i>s,X_j>t)-P(X_i>s)P(X_j>t)\le q(|i-j|)\alpha_{ij}(s,t),$$

for $s,t>0$,

$$P(X_i<s,X_j<t)-P(X_i<s)P(X_j<t)\le q(|i-j|)\beta_{ij}(s,t),$$

for $s,t<0$, where α_{ij} and β_{ij} are some nonnegative functions.

The above conditions are satisfied by sequences of pairwise independent, negatively quadrant dependent (NQD) and asymptotically quadrant independent (AQI) r.v.'s as well as by some sequences of mixing r.v.'s (cf. [1], [2]). For the sake of convenience let us recall definitions of some of these concepts of dependence. The AQI r.v.'s were introduced by Birkel (cf. [1]).

DEFINITION 2. A sequence $(X_n)_{n\in N}$ of r.v.'s is said to be AQI if the following conditions are satisfied:

$$|P(X_i>s,X_j>t)-P(X_i>s)P(X_j>t)|\le q(|i-j|)\alpha_{ij}(s,t),$$

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$$|P(X_i < s, X_j < t) - P(X_i < s)P(X_j < t)| \le q(|i-j|)\beta_{ij}(s,t),$$

for all $s, t \in R$ and $i \neq j$, where the sequence $(q(m))_{m\in N}$ and the functions α_{ij} and β_{ij} satisfy the same assumptions as in Definition 1.

The concept of quadrant dependence was introduced by Lehmann (cf. [8]).

DEFINITION 3. A sequence $(X_n)_{n\in N}$ of r.v.'s is pairwise NQD if

$$P(X_i > s, X_j > t) - P(X_i > s)P(X_j > t) \le 0,$$

or equivalently

$$P(X_i < s, X_j < t) - P(X_i < s)P(X_j < t) \le 0,$$

for all $s, t \in R$ and $i \neq j$. A sequence $(X_n)_{n\in N}$ is pairwise positively quadrant dependent (PQD) if the left–hand side in the above inequalities is nonnegative.

In recent days the bivariate dependence structure of the random variables is often described in terms of the copula function (cf. [10]). The aim of this note is to present some further examples of r.v.'s for which the AQSI notion seems to be particularly useful. By imposing some conditions on the copula we shall prove the strong law of large numbers for such sequences. Let us recall the definition of copula.

DEFINITION 4. Let X and Y be r.v.'s with distribution functions $F_X(x)$ and $F_Y(y)$, the function $C_{X,Y}(u,v)$ defined for $u, v \in [0,1]$ such that

$$P(X \le x, Y \le y) = C_{X,Y}(F_X(x), F_Y(y)) \tag{1}$$

is called the copula of X and Y.

By the Sklar's theorem this function is uniquely determined for $(u,v) \in Ran(F_X) \times Ran(F_Y)$ and it is well known that $C_{X,Y}(u,v)$ is a distribution function on $[0,1]\times[0,1]$ with uniform marginals (for details on copulas we refer the reader to [10]).

In this paper we shall consider sequences $(X_n)_{n\in N}$ of r.v.'s with copulas satisfying the following condition:

$$C_{X_i,X_j}(u,v) - uv \le \rho_{ij} uv(1-u)(1-v), \tag{2}$$

for $(u,v) \in Ran(F_{X_i}) \times Ran(F_{X_j})$ and $\rho_{ij} \ge 0$.

In the main result we shall also assume that ρ_{ij} depends only on $|i-j|$ in such a way that

$$\rho_{ij} := q(|i-j|) \to 0, \tag{3}$$

as $|i-j| \to \infty$, where q is some nonnegative function.

For sequences satisfying (2) we have

$$\begin{aligned} & P(X_i \le s, X_j \le t) - P(X_i \le s)P(X_j \le t) \\ \le\quad & q(|i-j|)P(X_i \le s)P(X_j \le t)P(X_i > s)P(X_j > t) \end{aligned}$$

furthermore it is easy to see that

$$\begin{aligned} & P\left(X_i > s, X_j > t\right) - P\left(X_i > s\right) P\left(X_j > t\right) \\ = \quad & P\left(X_i \le s, X_j \le t\right) - P\left(X_i \le s\right) P\left(X_j \le t\right). \end{aligned}$$

Thus, replacing s by $s - 1/n$ and t by $t - 1/n$ and letting $n \to \infty$ we see that such sequences are AQSI with

$$\begin{aligned} \alpha_{ij}(s,t) &= P\left(X_i \le s\right) P\left(X_j \le t\right) P\left(X_i > s\right) P\left(X_j > t\right) \\ \beta_{ij}(s,t) &= P\left(X_i < s\right) P\left(X_j < t\right) P\left(X_i \ge s\right) P\left(X_j \ge t\right) \end{aligned} \tag{4}$$

and $q(|i-j|) = \rho_{ij}$, provided (3) holds.

Let us observe that the condition (2) is satisfied by a fair number of important families of copulas. In the following examples we consider some sequences $(X_n)_{n\in N}$ of r.v.'s with the bivariate dependence structure described by a certain one–parameter family of copulas $C_{X_i,X_j}(u,v)$.

EXAMPLE 1. Farlie-Gumbel-Morgerstern copula $C_{X_i,X_j}(u,v) = uv(1+\theta_{ij}(1-u)(1-v))$, $-1 \le \theta_{ij} \le 1$ satisfies (2) with $\rho_{ij} = \theta_{ij} \vee 0$. For generalized FGM copula (cf. [7]) $C_{X_i,X_j}(u,v) = u^a v^a (1+\theta_{ij}(1-u)^b(1-v)^b)$, $a \ge 1$, $b \ge 1$, $0 \le \theta_{ij} \le 1$, we may put $\rho_{ij} = \theta_{ij}$.

EXAMPLE 2. Ali-Mikhail-Haq copula $C_{X_i,X_j}(u,v) = \frac{uv}{1-\theta_{ij}(1-u)(1-v)}$ satisfies (2) with $\rho_{ij} = 0$ for $\theta_{ij} \in [-1,0]$ and $\rho_{ij} = \frac{\theta_{ij}}{1-\theta_{ij}}$ for $\theta_{ij} \in (0,1)$.

EXAMPLE 3. The Plackett family of copulas is given by the equation

$$\theta_{ij} = \frac{C_{X_i,X_j}(u,v)(1-u-v+C_{X_i,X_j}(u,v))}{(u-C_{X_i,X_j}(u,v))(v-C_{X_i,X_j}(u,v))}$$

(cf. [10] for the explicit formula). Transforming this equation we get

$$C_{X_i,X_j}(u,v) - uv = (\theta_{ij}-1)\left(u - C_{X_i,X_j}(u,v)\right)\left(v - C_{X_i,X_j}(u,v)\right).$$

Thus, for $\theta_{ij} \in (0,1]$, (2) is satisfied with $\rho_{ij} = 0$. For $\theta_{ij} > 1$, noting that the Plackett family is positively ordered, we have $C_{X_i,X_j}(u,v) \ge uv$, so that $C_{X_i,X_j}(u,v) - uv \le (\theta_{ij}-1)uv(1-u)(1-v)$ and (2) is satisfied with $\rho_{ij} = \theta_{ij} - 1$.

If we impose some additional conditions on θ_{ij} in the above examples also (3) will be satisfied, for example if $\theta_{ij} \le 0$ in Example 1 and 2, then (3) holds trivially with $\rho_{ij} = q(|i-j|) \equiv 0$. Sequences of pairwise independent or pairwise NQD r.v.'s are trivial examples of sequences satisfying both conditions (2) and (3), a more informative example will be given in the next section.

2 Strong Law of Large Numbers

The classical Kolmogorov's strong law of large numbers for i.i.d. r.v.'s was generalized by Etemadi (cf. [5]) to sequences of pairwise independent r.v.'s and further by Matuła (cf. [9]) to pairwise NQD sequences. In the main result we shall generalize these SLLN's for AQSI sequences satisfying (2) and (3).

THEOREM 1. Let $(X_n)_{n\in N}$ be a sequence of identically distributed r.v.'s satisfying (2), (3) and such that $\sum_{m=1}^{\infty} q(m) < \infty$. Then, the following conditions are equivalent:

$$(X_1 + ... + X_n)/n \to a \tag{5}$$

almost surely for some constant a,

$$E|X_1| < \infty. \tag{6}$$

If $E|X_1| < \infty$, then $a = EX_1$.

PROOF. Let us observe that for α_{ij} and β_{ij}, defined by (4), we have

$$\begin{aligned}
\int_0^\infty \int_0^\infty \alpha_{ij}(t,s)dtds &\le \int_0^\infty \int_0^\infty P(X_i > s) P(X_j > t)\, dtds \\
&= EX_i^+ EX_j^+ \le (E|X_1|)^2 < \infty, \\
\int_{-\infty}^0 \int_{-\infty}^0 \beta_{ij}(t,s)dtds &\le \int_{-\infty}^0 \int_{-\infty}^0 P(X_i < s) P(X_j < t)\, dtds \\
&= EX_i^- EX_j^- \le (E|X_1|)^2 < \infty,
\end{aligned}$$

and the sufficiency of the condition $E|X_1| < \infty$ for the SLLN follows from Theorem 2.2 in [4]. In order to prove necessity let us observe that from the SLLN it follows that $X_n/n \to 0$ almost surely. We shall use the following version of the Borel-Cantelli lemma (cf. Theorem 8 in [2]). Let $(A_n)_{n\in N}$ be a sequence of events such that:

$$P(A_i \cap A_j) - P(A_i)P(A_j) \le q(|i-j|)P(A_j) \tag{7}$$

for all $i < j$. Assume that $\sum_{m=1}^{\infty} q(m) < \infty$. If $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $P(\limsup A_n) = 1$. Let us define $A_n = \{X_n > n\}$, for these events the assumption (7) is satisfied. Thus, if $\sum_{n=1}^{\infty} P(X_n > n) = \sum_{n=1}^{\infty} P(X_1 > n) = \infty$, then $P(\limsup A_n) = 1$, but it contradicts $X_n/n \to 0$ almost surely. Therefore $\sum_{n=1}^{\infty} P(X_1 > n) < \infty$. Similar considerations, for $A_n = \{X_n < -n\}$, yield $\sum_{n=1}^{\infty} P(X_1 < -n) < \infty$, so that we finally get $\sum_{n=1}^{\infty} P(|X_1| > n) < \infty$ which is equivalent to $E|X_1| < \infty$.

Now we shall give an example of an infinite sequence of r.v.'s satisfying the assumptions of our SLLN but not satisfying the SLLN neither of [5] nor [9].

EXAMPLE 4. Let us describe a sequence $(X_n)_{n\in N}$ of r.v.'s with the same distribution function $F(x)$ by introducing a consistent family of finite-dimensional FGM distributions (cf. [6]) in the following manner. The joint distribution of $X_{i_1}, ..., X_{i_n}$ is given by

$$F_{i_1,...,i_n}(x_1,...,x_n) = \prod_{k=1}^{n} F(x_k) \left(1 + \sum_{1\le j<k\le n} a_{i_j i_k}(1-F(x_j))(1-F(x_k))\right)$$

with $a_{i_j i_k} = \pm 2^{-i_j - i_k}$, since $\left|\sum_{1\le i<j\le k} a_{ij}\right| \le 1$ the choice of these constants is admissible i.e. the inequality (44.70) in [6] holds. The bivariate distribution of X_i, X_j is the FGM distribution with the copula as in Example 1 of the following form

$$C_{X_iX_j}(u,v) = uv(1 + a_{ij}(1-u)(1-v))$$

so that we may take $q(|i-j|) = 2^{-|i-j|}$ if $a_{ij} > 0$ and 0 otherwise. This sequence satisfies the conditions of our Theorem 1 obeying the SLLN iff $\int_{-\infty}^{\infty} |x| dF(x) < \infty$. Let us observe that the sign of a_{ij} describes whether the r.v.'s X_i and X_j are PQD or NQD therefore by taking all a_{ij} negative we obtain an infinite sequence of pairwise NQD r.v.'s for which the SLLN of [9] may be applied while the one of [5] not. For other choices of signs we get a sequence for which neither [5] nor [9] may be applied while our Theorem 1 holds.

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