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ABSTRACT DIFFERENCE EQUATIONS WITH NONLINEARITIES HAVING THE LOCAL LIPSCHITZ PROPERTIES [*†]

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Abstract

We consider semilinear difference equations in a Banach space. It is assumed that the nonlinearities of the considered equations satisfy the local Lipschitz condition. By virtue of the recent estimates for the norm of functions of quasi-Hermitian operators, explicit stability and boundedness conditions are given. Applications to infinite dimensional difference systems are discussed.

1 Introduction and Statement of the Main Result

Abstract nonlinear difference equations were considered in the papers by Agarwal et al. [1], Gil' [2], Gonzalez and Jimenez-Melado [5], Medina [6], Rodriguez [7], Rodriguez and Sweet [8], etc. But in these papers either boundary value problems were investigated or it is assumed that the nonlinear parts have the global Lipschitz properties.

In this paper, we consider the Cauchy problem for abstract nonlinear difference equations in a Banach space, whose nonlinear parts satisfy the local Lipschitz conditions. Solution estimates are established. They give us explicit stability and boundedness conditions. In Section 3, applications to infinite dimensional difference systems are also discussed. Our main tool in Section 3 is the recent estimates for the norm of functions of quasi-Hermitian operators.

Let X be a Banach space with norm $\|.\|_X$, and Y a Banach space with norm $\|.\|_Y$ imbedded in X; k_Y is the imbedding constant:

$$\|v\|_X \le k_Y \|v\|_Y \ \ v \in Y.$$

For a positive $r \le \infty$, put

$$\Omega_r(X) = \{h \in X : \|h\|_X \le r\}.$$

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Consider in X the nonlinear difference equation

$$x_{k+1} = Ax_k + F_k\left(x_k\right),\ k = 0, 1, 2, ..., \tag{1}$$

where A is a bounded linear operator in Y and $F_k : \Omega_r(X) \cap Y \to Y$ is continuous for each $k \geq 0$. In addition, there are nonnegative constants q and ξ such that

$$\left\|F_k\left(h\right)\right\|_Y \leq q\left\|h\right\|_Y + \xi,\ h \in \Omega_r(X); k = 0, 1, 2, ... \,. \tag{2}$$

THEOREM 1. Suppose the condition (2) holds. Suppose further that

$$M := \sup_j \left\|A^j\right\|_Y < \infty, \tag{3}$$

$$\theta := \sum_{j=0}^{\infty} \left\|A^j\right\|_Y < q^{-1} \tag{4}$$

and

$$k_Y \frac{M\left\|v\right\|_Y + \theta\xi}{1 - q\theta} < r,\ v \in Y. \tag{5}$$

Then any solution $\{x_k\}_{k=0}^{\infty}$ of (1) with the initial condition $x_0 = v$, satisfies the relations

$$\left\|x_k\right\|_Y \leq \frac{M\left\|v\right\|_Y + \theta\xi}{1 - q\theta},\ k = 1, 2, ... \,. \tag{6}$$

The proof of this theorem is presented in the next section.

2 Proof of the Theorem 1

We begin with a lemma.

LEMMA 1. Suppose condition (2) holds with $r = \infty$. Then, under conditions (3) and (4), any solution $\{x_k\}_{k=0}^{\infty}$ of (1), with the initial condition $x_0 = v$, satisfies inequality (6).

PROOF. It is well known that the general solution of (1) is given by

$$x_k = A^k x_0 + \sum_{j=0}^{k-1} A^{k-j-1} F_j\left(x_j\right),\ k = 1, 2, ... \,.$$

Hence,

$$\left\|x_k\right\|_Y \leq M\left\|x_0\right\|_Y + \sum_{j=0}^{k-1} \left\|A^{k-j-1} F_j\left(x_j\right)\right\|_Y .$$

Now, condition (2) implies

$$\|x_k\|_Y \le M\|x_0\|_Y + \sum_{j=0}^{k-1}\left\|A^{k-j-1}\right\|_Y \left(q\|x_j\|_Y + \xi\right).$$

Hence, it follows that

$$\begin{aligned}\sup_k \|x_k\|_Y &\le M\|v\|_Y + q\sup_k \|x_k\|_Y \sum_{j=0}^{k-1}\left\|A^{k-j-1}\right\|_Y + \xi\sum_{j=0}^{k-1}\left\|A^{k-j-1}\right\|_Y \\ &\le M\|v\|_Y + q\sup_k \|x_k\|_Y\ \theta + \xi\theta.\end{aligned}$$

Thus, condition (4) yields

$$\sup_k \|x_k\|_Y \le (1-q\theta)^{-1}[M\|v\|_Y + \xi\theta].$$

Hence the required result follows.

Proof of Theorem 1. If $r=\infty$, then the required result follows from Lemma 1. It remains to prove the statement in the case $r<\infty$. Define the functions

$$\widetilde{F}_k(x) = \begin{cases} F_k(x), \ \|x\|_X \le r \\ 0, \ \|x\|_X > r.\end{cases}$$

for $k=0,1,2,\dots$. Since

$$\left\|\widetilde{F}_k(v)\right\|_Y \le q\|v\|_Y + \xi,\ v\in Y; k=0,1,2,...,$$

then the sequence $\{\widetilde{x}_k\}_{k=0}^{\infty}$ defined by

$$\widetilde{x}_0 = x_0;\ \widetilde{x}_{k+1} = A\widetilde{x}_k + \widetilde{F}_k(x_k);\ k=1,2,...,$$

satisfies the inequality

$$\|\widetilde{x}_k\|_X \le k_Y\left(M\|x_0\|_Y + \theta\xi\right)(1-q\theta)^{-1} < r,$$

provided that (5) holds. But $F_k(x)$ and $\widetilde{F}_k(x)$ coincide on the ball $\Omega_r(X)$. From this we infer that $x_k = \widetilde{x}_k$ for $k=0,1,2,...$ and therefore (6) is satisfied, concluding the proof.

3 Infinite Dimensional Systems of Difference Equations

As usual, l^2 is the Hilbert space of number sequences equipped with the norm

$$\|h\|_{l^2} = \left[\sum_{k=1}^{\infty}|h_k|^2\right]^{\frac{1}{2}},\ h=(h_k)\in l^2$$

and c_0 is the Banach space of number sequences equipped with the norm

$$\|h\|_{c_0} = \sup_k |h_k|\,,\ h = (h_k) \in c_0.$$

Everywhere in this section

$$A = (a_{jk})_{j,k=1}^{\infty}$$

is an infinite matrix with real entries. It is assumed that

$$J(A) = \frac{1}{\sqrt{2}}[\sum_{j=1}^{\infty} \sum_{k=1, k\neq j}^{\infty} |a_{jk} - a_{kj}|^2]^{\frac{1}{2}} < \infty. \tag{7}$$

In this section, we take $Y = l^2$ and $X = c_0$ and $k_Y = 1$. So, (2) takes the form

$$\|F_k(h)\|_{l^2} \leq q\,\|h\|_{l^2} + \xi \quad (h \in \Omega_r(c_0))\,. \tag{8}$$

Assume that the spectral radius $\rho(A)$ of A satisfies the inequality

$$\rho(A) < 1. \tag{9}$$

Set

$$\psi(A) := \sup_m \sum_{k=0}^{m} \frac{m!\rho^{m-k}(A)\,J^k(A)}{(m-k)!\,(k!)^{\frac{3}{2}}}$$

and

$$\gamma(A) := \frac{\sqrt{2}}{1-\rho(A)} \exp\left(\frac{J^2(A)}{(1-\rho(A))^2}\right).$$

Below we check that under (9), $\psi(A) < \infty$.

THEOREM 2. Under conditions (7)-(9), let

$$q\gamma(A) < 1$$

and

$$\frac{\psi(A)\,\|v\|_{l^2} + \gamma(A)\,\xi}{1 - q\gamma(A)} < r,\ v \in l^2.$$

Then any solution $\{x_k\}_{k=0}^{\infty}$ of (1), with the initial condition $x_0 = v$, satisfies the relations

$$\|x_k\|_{l^2} \leq (\psi(A)\,\|v\|_{l^2} + \gamma(A)\,\xi)\,(1 - q\gamma(A))^{-1}\,,\ k = 1, 2, ... \,.$$

PROOF. We need the following result: under condition (7), the inequality

$$\|A^m\|_{l^2} \leq \sum_{k=0}^{m} \frac{m!\rho^{m-k}(A)\,g_I^k(A)}{(m-k)!\,(k!)^{\frac{3}{2}}},\ m = 1, 2, ...$$

is valid, where

$$g_I(A) := \left[2N^2(A_I) - 2\sum_{k=1}^{\infty} |\, Im\,(\lambda_k(A))|^2\right]^{\frac{1}{2}} \le J(A),$$

$A_I = \frac{A-A^*}{2i}$, $N(.)$ is the Hilbert-Schmidt norm and $\lambda_k(A)$ are the eigenvalues of A (see [4, Theorem 7.10.1]). Thus, $M \le \psi(A)$ and

$$\theta_l = \sum_{j=0}^{\infty} \left\|A^j\right\|_{l^2} \le \sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{j!\rho^{j-k}(A)\, g_I^k(A)}{(j-k)!\,(k!)^{\frac{3}{2}}}.$$

But

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{j!\rho^{j-k}(A)\, g_I^k(A)}{(j-k)!\,(k!)^{\frac{3}{2}}} &= \sum_{j=0}^{\infty} \frac{g_I^k(A)}{(k!)^{\frac{1}{2}}} \sum_{j=k}^{\infty} \frac{j!\rho^{j-k}(A)}{(j-k)!k!} \\ &= \sum_{j=0}^{\infty} \frac{g_I^k(A)}{(k!)^{\frac{1}{2}}} (1-\rho(A))^{-k-1}.\end{aligned}$$

Hence it follows that $\psi(A) < \infty$. By the Schwarz inequality we get

$$\sum_{j=0}^{\infty} \frac{g_I^k(A)}{(k!)^{\frac{1}{2}}} (1-\rho(A))^{-k-1} \le \sqrt{2}\,(1-\rho(A))^{-1} \exp\left[g_I^2(A)\,(1-\rho(A))^{-2}\right].$$

Thus, $\theta \le \gamma(A)$. Now by Theorem 1, we conclude that Theorem 2 is valid.

REMARK. Estimates for $\rho(A)$ can be found in [3].

Let us give an example of $A = (a_{jm})$ and F_k that satisfy conditions (7)-(9). Take

$$a_{jm} = \frac{1}{m(1+j)},\ j \le m,$$

and

$$a_{jm} = 0,\ j > m.$$

Then

$$\rho(A) = \sup_j |a_{jj}| = \sup_{j\ge 1} \frac{1}{j(1+j)} = \frac{1}{2}$$

cf. [3]. Moreover,

$$J^2(A) = \sum_{j=1}^{\infty}\sum_{k=j+1}^{\infty} \frac{1}{k^2(1+j)^2} < \infty.$$

So conditions (7) and (9) are satisfied. Furthermore, take

$$F_k(v) = (f_{kj}(v))_{j=1}^{\infty}$$

with

$$f_{kj}(v) = d_{jk} \sum_{m=1}^{\infty} v_m^p, \; k, j = 1, 2, ...; v = (v_k) \in l^2, p > 2,$$

where d_{jk} are real constants with

$$q_1 := \sup_k [\sum_{j=1}^{\infty} |d_{jk}|^2]^{1/2} < \infty.$$

Then

$$\|F_k(v)\|_{l^2}^2 \le \sum_{j=1}^{\infty} |d_{jk}|^2 [\sum_{m=1}^{\infty} |v_m|^p]^2 \le q_1^2 [r^{p-2} \sum_{m=1}^{\infty} |v_m|^2]^2, \; \|v\|_{c_0} \le r.$$

So condition (8) also holds with $q = q_0 r^{p-2}, \xi = 0$.

References

[1] R. P. Agarwal, H. B. Thompson and C. C. Tisdell, Difference equations in Banach spaces, Advances in difference equations IV, Comput. Math. Appl., 45(2003), 1437–1444.

[2] M. I. Gil', Periodic solutions of abstract difference equations, Applied Mathematics E-Notes, 1(2001), 18–23.

[3] M. I. Gil', Invertibility and spectrum of Hille-Tamarkin matrices, Math. Nachr. 244(2002), 78–88.

[4] M. I. Gil', Operator Functions and Localization of Spectra, Lectures Notes In Mathematics, Vol. 1830, Springer Verlag, Berlin, 2003.

[5] C. Gonzalez and A. Jimenez-Melado, An application of Krasnoselskii fixed point theorem to the asymptotic behavior of solutions of difference equations in Banach spaces, J. Math. Anal. Appl., 247(2000), 290–299.

[6] R. Medina, Solvability of discrete Volterra equations in weighted spaces, Dynamic Systems Appl., 5(1996), 407–422.

[7] J. Rodriguez, On nonlinear discrete boundary value problems, J. Math. Anal. Appl., 114(2)(1986), 398–408.

[8] J. Rodriguez and D. Sweet, Nonlinear boundary value problems on sequence spaces, J. Difference Eq. Appl., 8(2)(2002), 153–162.