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ANALYTICAL FORMULAS FOR SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS *

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Abstract

We present folmulas for general solutions of three terms linear difference equations with non-constant coefficients.

1 Introduction

The problem of obtaining analytical formulas for general solutions of difference equations has been studied by many authors, e.g. Agarwal [1], Bartoszewski and Kwapisz [4], Kwapisz [6]-[7], Lakshmikantham and Trigiante [8], Musielak and Popenda [9], Popenda [10], etc. In papers [1], [7] and [8], an explicit formula for the solution of the equation

$$F_{n+1} = a_n F_n + f_n, \quad n = 0, 1, \ldots,$$

is included.

In [4] and [6], the authors gave an analytical formula for the solutions of the equation

$$F_{n+1} = a_n F_n + b_n F_{\gamma_n} + f_n, \quad n = 0, 1, \ldots, \quad \gamma_n = n - \beta_n,$$

where β_n is the remainder obtained from dividing n by a fixed natural number k.

Popenda in [10] gave explicit formulas for the solutions of linear homogeneous second order equations

$$a_n x_{n+2} + b_n x_{n+1} + c_n x_n = 0.$$

Popenda and Musielak were also interested in the partial difference equation of the form

$$y(m+1, n+1) - y(m+1, n) - y(m, n+1) + y(m, n) = a(m, n) y(m, n).$$

They have presented in [9] the explicit formula for the solutions of the above equation. Popenda and Andruch-Sobiło considered the difference equations in groups [3].

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Andruch-Sobiło has also published some results on the difference equations in the groups in [2], some of the results in it are continuation of the work in [3].

The construction of the explicit formulas for the solutions of the partial difference equation is of great interest. For instance, Cheng has presented a lot of explicit solutions for partial difference equations in his book [5].

The problem of the explicit formulas for the solutions of difference equations is considered in this paper:

$$y(n+m) = a(n)y(n+1) + y(n), \tag{E1}$$

$$y(n+m) = a(n)[y(n+1) + dy(n)], \tag{E2}$$

$$y(n+m) = a(n)y(n+1) + b(n)y(n), \tag{E3}$$

where $a, b : N \to R\backslash\{0\}$, $d \in R$, $m \geq 2$, $n \in N$, are considered. The results contained in this note are continuation of the work began by Popenda (in [10])

Explicit formulas for general solutions of the above equations are presented. The analytical formulas are non-recurrent algorithms for obtaining solutions of (E1), (E2) and (E3).

2 The Sum Operators

To construct analytical formulas, 'sum operators' have to be defined. The symbol $\mathrm{mod}(w, z)$ denotes remainder of w/z, where $w, z \in Z$, $\mathrm{mod}(w, z)$ simplifies the equivalent expression $w - z\lfloor w/z \rfloor$, where the symbol $\lfloor w/z \rfloor$ denotes the greatest integral less than or equal to w/z.

DEFINITION 1. Let $a : N \to R\backslash\{0\}$, $m \in N$ and $n \in Z$. The operator $U(a; m, r, n) = 1$ if $r = 0$, and

$$\begin{aligned} U(a; m, r, n) \quad = \quad & \sum_{j_1=r-1}^{\lfloor (n+r-2)/m \rfloor} \sum_{j_2=r-2}^{j_1-1} \cdots \sum_{j_r=0}^{j_{r-1}-1} \{a(mj_1 - (r-2) + \mathrm{mod}(n+r-2, m)) \\ & \times a(mj_2 - (r-3) + \mathrm{mod}(n+r-2, m)) \\ & \times \cdots \\ & \times a(mj_r + 1 + \mathrm{mod}(n+r-2, m))\} \end{aligned}$$

if $r \geq 1$.

The operator U defined above is the sum of some products of the r-elements of sequence $\{a_n\}$. The value of the parameter r determines the number of elements in the products. For example, if $r = 1$ then in $U(a; m, r, n)$, there is only a simple sum. In particular,

$$U(a; 2, 1, 3) = \sum_{j_1=0}^{1} a(2j_1 + 1) = a(1) + a(3).$$

If $r = 2$ in $U(a; m, r, n)$, there are products of two terms. In particular,

$$U(a; 2, 2, 5) = \sum_{j_1=1}^{2} a(2j_1 + 1) \sum_{j_2=0}^{j_1-1} a(2j_2 + 2) = a(3)a(2) + a(5)a(2) + a(5)a(4).$$

For $r = 3$, products of three terms occur, an example is

$$\begin{aligned} U(a; 2, 3, 5) &= \sum_{j_1=2}^{3} a(2j_1 - 1) \sum_{j_2=1}^{j_1-1} a(2j_2) \sum_{j_3=0}^{j_2-1} a(2j_3 + 1) \\ &= a(3)a(2)a(1) + a(5)a(2)a(1) + a(5)a(4)a(1) + a(5)a(4)a(3). \end{aligned}$$

DEFINITION 2. Let $n \in Z$ and $m \in N$,

$$\rho^i(m, n) = \begin{cases} m\lfloor (n - i + 1)/(m - 1) \rfloor + i - n & for \quad i = 2, \ldots, m, \\ m\lfloor (n - m)/(m - 1) \rfloor + m + 1 - n & for \quad i = 1, m \geq 2, n \in Z, \end{cases}$$

and

$$\kappa^i(m, n) = \mathrm{mod}(\rho^i(m, n), m), \quad i = 1, 2, \ldots, m, \quad m \in N, \quad n \in Z, \quad m \geq 2.$$

COROLLARY 1. From Definition 1 the following properties of the operator U, for $r \geq 1$ and $m \geq 2$, can be observed:

$$U(a; m, r, n) = a(n)U(a; m, r - 1, n - m + 1)$$

if $\lfloor (n + r - 2)/m \rfloor = r - 1$, and

$$U(a; m, r, n) = U(a; m, r, n - m) + a(n)U(a; m, r - 1, n - m + 1)$$

if $\lfloor (n + r - 2)/m \rfloor > r - 1$, where $a : N \to R \backslash \{0\}$.

We will adopt the convention that $0^0 = 1$, $0^1 = 0$, empty sum is 0 and empty product is 1.

3 Main Results

Let

$$W(a; m, \kappa^i(m, n), n) := U(a; m, \kappa^i(m, n), n) + \sum_{j=1}^{\lfloor \rho^i(m,n)/m \rfloor} U(a; m, m \cdot j + \kappa^i(m, n), n).$$

THEOREM 1. The solution of (E1) satisfying initial conditions $y(i) = C_y(i)$, $i = 1, 2, \ldots, m$, $m \geq 2$, can be presented in the form

$$y(n + m) = \sum_{i=1}^{m} \left\{ W(a; m, \kappa^i(m, n), n) \right\} C_y(i), \quad n \in \{-m + 1, \ldots, 0\} \cup N. \qquad (1)$$

PROOF. The formula (1) for $n \in \{-m+1, \ldots, 0\}$ satisfies initial conditions, as follows. Let $n = 0$. For any $m \geq 2$,

$$\begin{aligned} y(m) &= \sum_{i=1}^{m} W(a; m, \kappa^i(m,0), 0) C_y(i) \\ &= \sum_{i=1}^{m} \left\{ U(a; m, \kappa^i(m,0), 0) + \sum_{j=1}^{\lfloor \rho^i(m,0)/m \rfloor} U(a; m, m \cdot j + \kappa^i(m,0), 0) \right\} C_y(i) \\ &= \sum_{i=1}^{m} U(a; m, \kappa^i(m,0), 0) C_y(i) \\ &= U(a; m, \kappa^1(m,0), 0) C_y(1) \\ &\quad + \sum_{i=2}^{m-1} U(a; m, \kappa^i(m,0), 0) C_y(i) + U(a; m, \kappa^m(m,0), 0) C_y(m). \end{aligned} \tag{2}$$

It is known that

$$U(a; m, \kappa^1(m,0), 0) = 0, \;\; m \geq 2$$

and

$$U(a; m, \kappa^i(m,0), 0) = 0, \quad 2 \leq i \leq m-1, \quad m \geq 3.$$

So equality (2) takes the form

$$y(m) = U(a; m, \kappa^m(m,0), 0) C_y(m) = U(a; m, 0, 0) C_y(m) = C_y(m).$$

For $n \in N$ and any $m \geq 2$, the solution of difference equation is rewritten in the form

$$y(n+m) = \varphi_1(n) C_y(1) + \varphi_2(n) C_y(2) + \ldots \varphi_m(n) C_y(m).$$

By putting the above dependence to (E1) we obtain

$$\varphi_i(n) = a(n) \varphi_i(n+1-m) + \varphi_i(n-m), \quad n \in N \tag{3}$$

for each $i = 1, 2, \ldots, m$ separately. If $\varphi_1(n), ..., \varphi_m(n)$ are given by $W(a; m, \kappa^i(m,n), n)$, equation (3) can be presented in an explicit form:

$$\begin{aligned} W(a; m, \kappa^i(m,n), n) &= a(n) W(a; m, \kappa^i(m, n-m+1), n-m+1) \\ &\quad + W(a; m, \kappa^i(m, n-m), n-m). \end{aligned} \tag{4}$$

It is necessary to use Definitions 1 and 2 and properties of U in to prove the formula (4).

THEOREM 2. The solution of (E2) satisfying initial conditions $y(i) = C_y(i)$, $i = 1, 2, \ldots, m$, $m \geq 2$, can be presented in the form

$$y(n+m) = d^{(-m+1)(\lfloor (n+m-1)/m \rfloor)+(n+m)} \left\{ \prod_{j=0}^{\lfloor (n+m)/m \rfloor -1} a(j \cdot m + \mathrm{mod}(n+m, m)) \right\} \times \sum_{i=1}^{m} \{W(s; m, \kappa^i(m,n), n)\} d^{-i} C_y(i), \tag{5}$$

for $n \in \{-m+1, \ldots, 0\} \cup N$, where $s(0) = 1$,

$$s(n) = d^{(-m+1)\delta(n,m)} \prod_{i=0}^{\lfloor n/m \rfloor -1} \frac{a(i \cdot m + \mathrm{mod}(n,m) + 1)}{a(i \cdot m + \mathrm{mod}(n,m))}, \tag{6}$$

and

$$\delta(n,m) = \begin{cases} 1 & \mathrm{mod}(n,m) = 0 \\ 0 & \mathrm{mod}(n,m) \neq 0 \end{cases}.$$

PROOF. By putting

$$y(n) = d^n x(n), \quad n \in N, d \in R, \tag{7}$$

equation (E2) can be transformed to the form

$$x(n+m) = a(n) d^{-m+1} [x(n+1) + x(n)],\ n \in N. \tag{8}$$

The solution of (8) can be presented in the form $x(n) = u(n)v(n)$ where

$$u(n) = d^{(-m+1)(\lfloor (n-1)/m \rfloor)} \prod_{j=0}^{\lfloor n/m \rfloor -1} a(j \cdot m + \mathrm{mod}(n,m)),\ n \in N, \tag{9}$$

and $v(n)$ is the solution of the following equation

$$v(n+m) = s(n)v(n+1) + v(n),\ n \in N, \tag{10}$$

for $v(i) = C_v(i)$, $i = 1, 2, \ldots, m$.

In (6) and (9), it is assumed $a(0) = 1$. The equation (10) has the form (E1). The solution of (10) can be presented by the use of operator $U(a; m, r, n)$. Therefore the general solution of the equation (8) is given by the formula

$$x(n+m) = d^{(-m+1)(\lfloor (n+m-1)/m \rfloor)} \left\{ \prod_{j=0}^{\lfloor (n+m)/m \rfloor -1} a(j \cdot m + \mathrm{mod}(n+m, m)) \right\} \times \sum_{i=1}^{m} \{W(s; m, \kappa^i(m,n), n)\} C_v(i),$$

for $n \in \{-m+1,\dots,0\} \cup N$. Considering (7) the general solution of (E2) is obtained in form (5).

REMARK 1. For $d = 1$, (E2) is of the form $y(n+m) = a(n)[y(n+1) + y(n)]$.

THEOREM 3. The solution of $y(n+m) = a(n)[y(n+1) + y(n)]$ satisfying the initial conditions $y(i) = C_y(i)$, $i = 1, 2, \dots, m$, $m \geq 2$, can be presented in the form

$$y(n+m) = \left\{ \prod_{j=0}^{\lfloor (n+m)/m \rfloor - 1} a(j \cdot m + \operatorname{mod}(n+m, m)) \right\} \times \\ \times \sum_{i=1}^{m} \left\{ W(s; m, \kappa^i(m,n), n) \right\} C_y(i),$$

for $n \in \{-m+1,\dots,0\} \cup N$.

REMARK 2. For $d = -1$, (E2) takes the form $y(n+m) = a(n)\Delta y(n)$.

THEOREM 4. The solution of $y(n+m) = a(n)\Delta y(n)$ satisfying the initial conditions $y(i) = C_y(i)$, $i = 1, 2, \dots, m$, $m \geq 2$, can be presented in the form

$$y(n+m) = (-1)^{(-m+1)(\lfloor (n+m-1)/m \rfloor)+(n+m)} \prod_{j=0}^{\lfloor (n+m)/m \rfloor - 1} a(j \cdot m + \operatorname{mod}(n+m, m)) \\ \times \sum_{i=1}^{m} \left\{ W(s; m, \kappa^i(m,n), n) \right\} (-1)^{-i} C_y(i),$$

for $n \in \{-m+1,\dots,0\} \cup N$.

The proofs of Theorem 3 and Theorem 4 follow from the proof of the Theorem 2.

THEOREM 5. The solution of (E3) satisfying the initial conditions $y(i) = C_y(i)$, $i = 1, 2, \dots, m$, $m \geq 2$, can be presented in the form

$$y(n+m) = \prod_{j=1}^{n+m-1} \frac{b(j)}{a(j)} \left\{ \prod_{i=0}^{\lfloor (n+m)/m \rfloor - 1} d(i \cdot m + \operatorname{mod}(n+m, m)) \right\} \times \\ \times \left[\sum_{i=1}^{m} \{ W(s; m, \kappa^i(m,n), n) \} \prod_{k=0}^{i-1} \frac{a(k)}{b(k)} C_y(i) \right]$$

for $n \in \{-m+1,\dots,0\} \cup N$, where

$$\begin{aligned} d(n) &= b(n) \prod_{j=n}^{n+m-1} \frac{a(j)}{b(j)}, \ a(0) := 1, \ b(0) := 1, \ d(0) := 1, \\ s(n) &= \prod_{i=0}^{\lfloor n/m \rfloor - 1} \frac{d(i \cdot m + \operatorname{mod}(n,m) + 1)}{d(i \cdot m + \operatorname{mod}(n,m))}, \ s(0) := 1, \ n \in N. \end{aligned} \tag{11}$$

PROOF. By (11) and

$$z(n) = y(n) \prod_{j=1}^{n-1} \frac{a(j)}{b(j)}, \quad n \in N,$$

(E3) can be transformed to the form

$$z(n+m) = d(n)[z(n+1) + z(n)], \ n \in N. \tag{12}$$

The above equation is of the form (E2), therefore, if the formula for equation (12) is known the formula for the general solution of (E3) can be derived.

4 Examples

We offer some examples.

Examples of values of $\rho^i(4,n)$, $i = 1, 2, 3, 4$:

n	$\rho^1(4,n)$	$\rho^2(4,n)$	$\rho^3(4,n)$	$\rho^4(4,n)$
1	0	1	-2	-1
2	-1	0	1	-2
3	-2	-1	0	1
4	1	2	-1	0
5	0	1	2	-1
6	-1	0	1	2
7	2	3	0	1
8	1	2	3	0

Examples of values of $\kappa^i(4,n)$, $i = 1, 2, 3, 4$:

n	$\kappa^1(4,n)$	$\kappa^2(4,n)$	$\kappa^3(4,n)$	$\kappa^4(4,n)$
1	0	1	2	3
2	3	0	1	2
5	0	1	2	3
8	1	2	3	0

The first values of $U(a; 4, 1, n)$:

n	$U(a; 4, 1, n)$
$1, 2, 3, 4$	properly $a(1)$, $a(2)$, $a(3)$, $a(4)$
$5, 6, 7, 8$	properly $a(1) + a(5)$, $a(2) + a(6), \ldots,$ $a(4) + a(8)$
$9, \ldots, 12$	$a(1) + a(5) + a(9), \ldots,$ $a(4) + a(8) + a(12)$
13	$a(1) + a(5) + a(9) + a(13)$

The first values of $U(a; 4, 2, n)$:

n	$U(a; 4, 2, n)$
$1, 2, 3$	0
$4, 5, 6, 7$	properly $a(4)a(1)$, $a(5)a(2)$, $a(6)a(3)$, $a(7)a(4)$
$8, 9, 10, 11$	$a(4)a(1) + a(8)a(1) + a(8)a(5), \ldots, a(7)a(4) + a(11)a(4) + a(11)a(8)$

The first values of $U(a;4,3,n)$:

n	$U(a;4,3,n)$
$1,2,\ldots,6$	0
$7,8,9,10$	properly $a(7)a(4)a(1)$, $a(8)a(5)a(2),\ldots,$ $a(10)a(7)a(4)$,
11	$a(7)a(4)a(1)+a(11)a(4)a(1)+a(11)a(8)a(1)+a(11)a(8)a(5)$
12	$a(8)a(5)a(2)+a(12)a(5)a(2)+a(12)a(9)a(2)+a(12)a(9)a(6)$

The first values of $U(a;4,4,n)$:

n	$U(a;4,4,n)$
$1,2,\ldots,9$	0
$10,11,\ldots,13$	$a(10)a(7)a(4)a(1),\ldots,$ $a(13)a(10)a(7)a(4)$
14	$a(14)a(11)a(8)a(5)+a(14)a(11)a(8)a(1)+a(14)a(11)a(4)a(1)+$ $+a(14)a(7)a(4)a(1)+a(10)a(7)a(4)a(1)$

The solution of (E1) for $m=4$ by the formula (1) is of the form

$$y(n+4)=\sum_{i=1}^{4}\left\{U(a;4,\kappa^i(4,n),n)+\sum_{j=1}^{\lfloor\rho^i(4,n)/4\rfloor}U(a;4,4j+\kappa^i(4,n),n)\right\}C_y(i).$$

So, when $n=5$,

$$\begin{aligned}y(9)&=U(a;4,2,5)C_y(3)+U(a;4,1,5)C_y(2)+U(a;4,0,5)C_y(1)\\&=\{a(2)a(5)\}C_y(3)+\{a(5)+a(1)\}C_y(2)+C_y(1).\end{aligned}$$

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