Applied Mathematics E-Notes, 5(2005), 194-201 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

EXACT REGION OF STABILITY FOR AN INVESTMENT PLAN WITH THREE PARAMETERS*

Sui Sun Cheng†, Yi-Zhong Lin‡

Received 10 August 2004

Abstract

Necessary and sufficient conditions for the asymptotic stability of a class of difference equations with three parameters are obtained. These conditions are expressed in terms of subsets of the parameter space.

1 Introduction

A principal u_0 is invested for k interest periods, where k is a positive integer, at an effective rate of interest r per period. If we denote the accumulated principal at the end of n interest periods $(n = 1, 2, ..., k)$ by u_n and consider the growth in the n-th interest period, we obtain the simple difference equation

$$u_{n+1} - u_n = ru_n,\ n \in N = \{0, 1, 2, ..., k, ...\}. \tag{1}$$

Since the above equation has the simple solution

$$u_n = u_0(1 + r)^n,\ n \in N,$$

much can be said about its quantitative as well as the qualitative behavior. In contrast, if the original principal is divided into different parts and invested in financial instruments that may regenerate and/or take different time periods to yield interests, the corresponding difference equations may have solutions which are too complicated to analyze. In such cases, alternate means are necessary in order to gain insight into the nature of the investment policies.

In this paper, we will interpret u_n as the wealth of a person at the end of the time period n. Here the wealth is measured in real monetary units and may take on negative value if his liabilities are greater than his assets. To demonstrate the type of mathematical means mentioned above, we suppose he engages in investments so that his increase in wealth is governed by an equation of the form

$$u_{n+1} - u_n = au_{n-2} + bu_{n-1} + cu_n,\ n \in N = \{0, 1, 2, ...\}, \tag{2}$$

*Mathematics Subject Classifications: 91B28, 39A11

†Department of Mathematics, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. China.
‡Department of Mathematics, Fujian Normal University, Fuzhou, Fujian 350007, P. R. China

where $a, b, c \in R$. We remark that the numbers a, b, c can be negative to signal wrong investment operations. For similar difference equations that arise in economic dynamics, the book [1] by Gandolfo can be consulted.

Since (2) is a recurrence relation, given fixed real numbers u_{-2}, u_{-1} and u_0, it is then easy to calculate from it in a recursive manner the subsequent terms $u_1, u_2, \ldots$. The sequence $\{u_i\}_{i=-2}^{\infty}$ is called a solution of (2).

This equation (2) is linear, homogeneous and has constant coefficients. It is therefore possible to find the set of all solutions of equation (2) by finding the roots of the characteristic polynomial [1, Chapter 6]

$$f(\lambda|a, b, c) = \lambda^3 - (c+1)\lambda^2 - b\lambda - a,$$

and qualitative behavior of equation (2) can then be obtained by analyzing the corresponding solution space. However, it is well known, that although a third order polynomial such as $f(\lambda|a, b, c)$ can be solved systemically in principle, the solutions can be very complicated. Besides, these solutions will depend on the parameters a, b and c, which will most likely cause additional difficulties.

Among the qualitative properties of equation (2), a particularly important one is its asymptotic stability. More precisely, we say that equation (2) is (globally) asymptotically stable if each of its solution converges to 0. Since a, b and c are not given explicitly in (2), asymptotic stability may vary as these parameters change. For this reason, we will treat (a, b, c) as a point in the Euclidean space R^3, and try to find the set Ω of all points in R^3 such that for each point in this subset Ω, the corresponding equation (2) is asymptotically stable.

Before the formal analysis, we first recall a well known result [1] that asserts that a linear homogeneous difference equation with constant coefficients such as (2) are asymptotically stable if, and only if, all roots of its characteristic polynomial is subnormal. Here a root z of a real polynomial is subnormal if $|z| < 1$, normal if $|z| = 1$ and supernormal if $|z| > 1$. Therefore Ω is the set of all points in R^3 such that for each point (a, b, c) in it, the corresponding $f(\lambda|a, b, c)$ has subnormal roots only.

The region of stability Ω has been found by considering the relations of the roots of (1) and its coefficients [2, pp. 327-328]. Here we present an alternate approach based on continuity and simple geometric arguments.

2 Bounding Regions

For the sake of convenience, we will denote the maximum of the moduli of the roots of $f(\lambda|x, y, z)$ by $\rho(x, y, z)$. It is well known that $\rho(x, y, z)$ is a continuous function with respect to (x, y, z). Therefore, the region of stability Ω is open and its boundary is contained in the set of points (x, y, z) such that $f(\lambda|x, y, z)$ has a normal root. For this reason, let us first consider the case when $f(1|x, y, z) = 0$ and $f(-1|x, y, z) = 0$ as well as $f(e^{\pm i\theta}|x, y, z) = 0$ where $\theta \in (0, \pi)$. The first case leads to

$$f(1|x, y, z) = -(x + y + z) = 0, \tag{3}$$

while the second leads to

$$f(-1|x, y, z) = y - z - x - 2 = 0, \tag{4}$$

and the third leads to the system

$$\cos 3\theta - (z+1)\cos 2\theta - y\cos\theta - x = 0, \quad (5)$$

$$\sin 3\theta - (z+1)\sin 2\theta - y\sin\theta = 0, \quad (6)$$

where $\theta \in (0,\pi)$. Since equation (6) can be rewritten as

$$y = \frac{\sin 3\theta - (z+1)\sin 2\theta}{\sin\theta},$$

equation (5) can then be rewritten as

$$x = \frac{(z+1)\sin\theta - \sin 2\theta}{\sin\theta} = (z+1) - 2\cos\theta. \quad (7)$$

Since $\theta \in (0,\pi)$, thus $z-1 < x < z+3$ and

$$y = -1 + 4\cos^2\theta - (z+1)2\cos\theta = x^2 - (z+1)x - 1, \ x \in (z-1, z+3). \quad (8)$$

The surface in R^3 defined by (3) separates R^3 into two parts: $x+y+z>0$ and $x+y+z<0$. We assert that Ω is contained in the latter subset. To see this, note that $\lim_{\lambda\in R,\lambda\to\infty} f(\lambda|x,y,z) = +\infty$ and $\lim_{\lambda\in R,\lambda\to-\infty} f(\lambda|x,y,z) = -\infty$. If $a+b+c \geq 0$, then $f(1|a,b,c) \leq 0$. Thus there exists a real root $\lambda_* \geq 1$ such that $f(\lambda_*|a,b,c) = 0$. This is contrary to the definition of Ω. Similarly, we can show that Ω is contained in the region $\left\{(x,y,z)\in R^3 |\ -x+y-z-2<0\right\}$.

The following is now clear.

LEMMA 1. The region of stability Ω is contained in the set

$$\Gamma = \left\{(x,y,z)\in R^3 |\ x+y+z<0, -x+y-z-2<0\right\},$$

and the set of points $(x,y,z)\in R^3$ such that $f(\lambda|x,y,z)$ has a normal root is contained in

$$\left\{(x,y,z)\in R^3 |\ x+y+z=0\right\}$$

$$\left\{(x,y,z)\in R^3 |\ -x+y-z-2=0\right\}$$

or

$$\left\{(x,y,z)\in R^3 |\ y = x^2-(z+1)x-1, z-1<x<z+3\right\}.$$

3 Region of Stability

In order to visualize the three dimensional region of stability Ω, we will consider its level sets at each given $z=c$. To this end, we will denote such a level set by Ω_c, that is,

$$\Omega_c = \left\{(x,y)\in R^2 |\ (x,y,c)\in\Omega\right\}.$$

We will also denote the level set of Γ at the point $z = c$ by Γ_c, that is,

$$\Gamma_c = \left\{(x, y) \in R^2 |\ x + y + c < 0, -x + y - c - 2 < 0\right\}.$$

Note that (8) can be rewritten as

$$y = p(x) \equiv \left(x - \frac{z+1}{2}\right)^2 - \frac{1}{4}\left\{4 + (z+1)^2\right\},\ x \in (z-1, z+3).$$

We will denote its graph by P, and the level set of P at $z = c$ by P_c. Note further that P_c is part of a parabola in the x, y-plane.

THEOREM 1. The region of stability Ω is contained in $\left\{(x, y, z) \in R^3 |\ z < 2\right\}$.

Proof. We first show that if $c \geq 2$, P_c is outside the region Γ_c. Indeed, note that when $c \geq 2$, Γ_c is just the set of points $(x, y) \in R^2$ which satisfies $y < -x - c$ and $y < x + c + 2$, that is, $y < x + c + 2$ for $x \leq -c - 1$ and $y < -x - c$ for $x \geq -c - 1$. Therefore, for $x \in (c-1, c+3)$, the function that describes the bounding line segment of this set is given by

$$h(x) = -x - c,\ c - 1 < x < c + 3.$$

Since

$$p(c-1) - h(c-1) = 0,$$

$$p'(c-1) - h'(c-1) = c - 2,$$

and

$$p''(c-1) - h''(c-1) = 2,$$

thus if $c - 2 \geq 0$, then P_c will be strictly above the line segment defined by $h(x)$ for $c - 1 < x < c + 3$. In view of Lemma 1, we have shown that $f(\lambda|a, b, c)$ does not have any normal roots for any $(a, b) \in \Gamma_c$.

On the other hand, if we pick $x = 0$ and $y = -(c+1)^2/4$ where $c \geq 2$, then $(x, y) \in \Gamma_c$. Furthermore, since

$$f\left(\lambda|0, -(c-1)^2/4, c\right) = \lambda\left(\lambda - \frac{c+1}{2}\right)^2 = 0$$

has simple root $\lambda = 0$ and double root $\lambda = (c+1)/2$, thus $\rho(0, -(c-1)^2/4, c) > 1$.

We now assert that if $c \geq 2$, $\rho(x, y, c) > 1$ for any $(x, y) \in \Gamma_c$. Indeed, it this is not the case, then there is some $(x_0, y_0) \in \Gamma_c$ such that $\rho(x_0, y_0, c) < 1$. If we now connect the points $(0, -2(c+1)^2/4)$ and (x_0, y_0) by a continuous curve completely contained inside Γ_c (which can be done in view of the special form of Γ_c), then by continuity, there would be some point (x^*, y^*) on this curve such that $\rho(x^*, y^*, c) = 1$. But then $f(\lambda|x^*, y^*, c)$ has a normal root. This is contrary to what we have shown above. The proof is complete.

The above result and the next can be proved in a very simple manner. Indeed, since $|z+1|$ is equal to the absolute value of the sum of the three roots of $f(\lambda|x,y,z)$, thus $|z+1|<3$ is a necessary condition for all roots of $f(\lambda|x,y,z)$ to be subnormal. However, we remark that the last part of the above proof makes use of the pathwise connectedness property of the region in concern and continuity arguments. Similar ideas can also be used again several times in the following discussions. In particular, we may show the following result. Since the proof is similar to that above, it will only be sketched.

THEOREM 2. The region of stability Ω is contained in $\left\{(x,y,z)\in R^3|\ z>-4\right\}$.

Sketch of Proof. We first show that if $c\le -4$, P_c is outside the region Γ_c. This is shown by noting that the corresponding Γ_c is just the set of points $(x,y)\in R^2$ which satisfies $y<-x-c$ and $y<x+c+2$ so that function the describes the bounding boundary for $x\in(c-1,c+3)$ is given by

$$h(x)=x+c+2,\ c-1<x<c+3.$$

Comparing the function $p(x)$ and $h(x)$, we see that if $c\le -4$, then the cross section P_c will be strictly above the line segment defined by $h(x)$ for $c-1<x<c+3$. In other words, $f(\lambda|a,b,c)$ does not have any normal roots for any $(a,b)\in\Gamma_c$. On the other hand, we can pick $(u,v)\in\Gamma_c$ such that $f(\lambda|u,v,c)$ has a real supernormal root λ^*. Finally, as in the proof of Theorem 1, we may show that $\rho(x,y,c)\ge 1$ for any $(x,y)\in\Gamma_c$ by continuity arguments. The proof is complete.

After we have shown that Ω is bounded between the planes $z=-4$ and $z=2$, we may now consider three different cases; (i) $0\le c<2$, (ii) $-2<c<0$, and (iii) $-4<c\le -2$.

THEOREM 3. For $0\le c<2$, the corresponding level set Ω_c is equal to

$$A_c=\left\{(x,y)\in R^2|\ c-1<x<1, x^2-(c+1)x-1<y<-x-c\right\}. \tag{9}$$

Proof. For each $c\in[0,2)$, Γ_c is given by the set of points (x,y) that satisfy $y<x+c+2$ for $x\le -c-1$ and $y<-x-c$ for $x\ge -c-1$. It is not difficult to verify that the parabola defined by

$$\tilde{p}(x)=x^2-(c+1)x-1,\ x\in R,$$

intersects with the straight line segment

$$h(x)=-x-c,\ x\ge -c-1 \tag{10}$$

at $(x,y)=((c-1),1-2c)$ and $(x,y)=(1,-(c+1))$, but does not intersect the straight line segment $q(x)=x+c+2$ over $x\le -c-1$. Thus, the part of P_c for $x\in(c-1,1)$ lies inside Γ_c and separates Γ_c into two disjoint open and pathwise connected regions one of which is given by the set A_c defined by (9). For the sake of convenience, let us denote the other region by B_c. We will show that there is a point (u_2,v_2) in B_c such that $f(\lambda|u_2,v_2,c)$ has a supernormal root, and a point (u_1,v_1) in A_c such that all roots of $f(\lambda|u_1, v_1, c)$ are subormal. To see this, let $(u_2, v_2) = (0, -5)$. Then it is easily seen that (u_2, v_2) is in B_c. Furthermore,

$$f(\lambda|u_2, v_2, c) = \lambda\left\{\lambda^2 - (c+1)\lambda + 5\right\},$$

so that its roots are 0, $\frac{1}{2}\left\{c+1 \pm i\sqrt{20-(c+1)^2}\right\}$ and therefore the corresponding $\rho(u_2, v_2, c) = \sqrt{5} > 1$. Next, if $c \in [0, 1)$, let $(u_1, v_1) = (0, -(c+1)/2)$ and if $c \in [1, 2)$, we let

$$(u_1, v_1) = \left(\left(\frac{c+1}{3}\right)^3, -\frac{1}{3}(c+1)^2\right).$$

It is easy to see that $(0, -(c+1)/2) \in A_c$ if $c \in [0, 1)$. Furthermore, the corresponding

$$f(\lambda|u_1, v_1, c) = \lambda\left\{\lambda^2 - (c+1)\lambda + \frac{c+1}{2}\right\}$$

has roots 0 and $\frac{1}{2}\left\{c+1 \pm i\sqrt{1-c^2}\right\}$. Therefore the corresponding $\rho(u_1, v_1, c) = (c+1)/2 < 1$.

To see that $(u_1, v_1) \in A_c$ for $c \in [1, 2)$, first note that $c < 2$ implies $(c+1)/3 < 1$ and $(c+1)^3/3^3 < 1$. Furthermore, since the function

$$g(x) = \left(\frac{x+1}{3}\right)^3 - (x-1)$$

is strictly decreasing on $[1, 2)$ and $g(2) = 0$, we see that $(c-1) < (c+1)^3/3^3$ for $1 \le c < 2$. We have thus shown that $c - 1 < u_1 < 1$. Next, consider

$$w(x) = -x - \left(\frac{x+1}{3}\right)^3 - \left(-\frac{(x+1)^2}{3}\right), \ 1 \le x < 2.$$

Since $w(2) = 0$ and

$$w'(x) = -\frac{(x-2)^2}{9} < 0, \ 1 \le x < 2,$$

we have $w(x) > 0$ for $x \in [1, 2)$ so that

$$-\frac{1}{3}(c+1)^2 < -c - \left(\frac{c+1}{3}\right)^3, \ 1 \le c < 2.$$

Next, consider

$$q(x) = -\frac{1}{3}(x+1)^2 - \left\{\left(\frac{x+1}{3}\right)^6 - (x+1)\left(\frac{x+1}{3}\right)^3 - 1\right\}, \ 1 \le x < 2.$$

Since $q(2) = 0$ and

$$q'(x) = -\frac{2}{3}(x+1) - 2\left(\frac{x+1}{3}\right)^5 + 4\left(\frac{x+1}{3}\right)^3,$$

$$q''(x) = -\frac{2}{3} - \frac{10}{3}\left(\frac{x+1}{3}\right)^4 + 4\left(\frac{x+1}{3}\right)^2,$$

so $q'(2) = 0$, $q''(1) = 110/243 > 0$ and $q''(2) = 0$. Since

$$q'''(x) = -\frac{8}{27}\frac{x+1}{9}\left(5x^2 + 10x + 22\right),$$

we see further that $q'''(1) > 0$, $q'''(2) < 0$ and $q'''(x)$ has a unique root in $(1, 2]$. Thus $q''(x) > 0$ for $1 \le x < 2$ which implies $q(x)$ is strictly convex over $(1, 2]$ and $q(x) > 0$ for $x \in (1, 2)$. Thus

$$\left(\frac{c+1}{3}\right)^6 - (c+1)\left(\frac{c+1}{3}\right)^3 - 1 < -\frac{(c+1)^2}{3},\ 1 \le c < 2.$$

In view of these inequalities, $(u_1, v_1) \in A_c$. Since the corresponding characteristic polynomial is

$$f(\lambda|u_1, v_1, c) = \left(\lambda - \frac{c+1}{3}\right)^3,$$

we see that $\rho(u_1, v_1, c) = (c+1)/3 < 1$ for $1 \le c < 2$.

By means of the continuity arguments shown in the proof of Theorem 1, we may now assert that for every $(x, y) \in A_c$, the corresponding $\rho(x, y, c) < 1$ and for every (x, y) in the complement $\Gamma_c \backslash A_c$, the corresponding $\rho(x, y, c) \ge 1$. In other words, $A_c = \Omega_c$. The proof is complete.

THEOREM 4. For $-4 < c \le -2$, the corresponding level set Ω_c is equal to

$$D_c = \left\{(x, y) \in R^2 |\ -1 < x < c+3, x^2 - (c+1)x - 1 < y < x + c + 2\right\}. \tag{11}$$

Proof. The proof is similar to that of Theorem 3 and will thus be sketched. For each $c \in (-4, -2]$, Γ_c is given by the set of points (x, y) that satisfy $y < x + c + 2$ for $x \le -c - 1$ and $y < -x - c$ for $x \ge -c - 1$. It is not difficult to verify that the parabola defined by

$$\tilde{p}(x) = x^2 - (c+1)x - 1,\ x \in R,$$

intersects with the straight line segment

$$q(x) = x + c + 2,\ x \le -c - 1 \tag{12}$$

at $(x, y) = (-1, c+1)$ and $(x, y) = (c+3, 2c+5)$, but does not intersect the straight line segment $h(x) = x + c + 2$ over $x \ge -c - 1$. Thus the part of P_c for $x \in (-1, c+3)$ lies inside Γ_c and separates Γ_c into two disjoint open and pathwise connected regions one of which is given by the set D_c defined by (11). Next, we may show that $(0, -5)$ belongs to the complement of $\overline{D}_c$ relative to Γ_c and the corresponding characteristic polynomial $f(\lambda|0, -5, c)$ has supernomal roots. We may also show that for $c \in (-3, -2]$, the point $(0, (c-1)/4)$ belongs to D_c, and for $c \in (-4, -3]$, the point $\left((c+1)^3/3^3, -(c+1)^2/3\right)$

belongs to D_c. Furthermore, in both cases, the roots of the corresponding characteristic polynomial are all subnormal. By continuity arguments similar to that in the proof of Lemma 2, we may then assert that for each $(x, y) \in D_c$, the corresponding $\rho(x, y, c) < 1$ and for each (x, y) in the complement $\Gamma_c \backslash D_c$, the corresponding $\rho(x, y, c) \geq 1$. The proof is complete.

THEOREM 5. For $-2 < c < 0$, the corresponding level set Ω_c is equal to

$$E_c = \left\{(x, y) \in R^2 | \ |x| < 1, x^2 - (c+1)x - 1 < y < \min\{-x - c, c + x + 2\}\right\}. \quad (13)$$

Proof. The proof is similar to that of Theorem 3 and will be sketched. First we may show that, for each $c \in (-2, 0)$, the parabola defined by $\tilde{p}(x) = x^2 - (c+1)x - 1$ for $x \in R$ intersects the straight line segment defined by (12) at $(-1, c+1)$, and intersects the straight line segment defined by (10) at $(1, -(c+1))$. Therefore the part of P_c for $x \in (-1, 1)$ is inside Γ_c and separates it into two disjoint open regions one of which is given by the set E_c defined by (13). Next, we may easily see that $(0, 0)$ belongs to E_c and the corresponding $\rho(0, 0, c) < 1$. We may also show that the point $(0, -5)$ belongs to the complement $\Gamma_c \backslash \overline{E}_c$ and the corresponding $\rho(0, -5, c) > 1$. By continuity arguments, we may then assert that for each $(x, y) \in E_c$, $\rho(x, y, c) < 1$ and for each $(x, y) \in \Gamma_c \backslash E_c$, $\rho(x, y, c) \geq 1$. The proof is complete.

We may summarize our results in the following form: The roots of the real polynomial $z^3 - c'z^2 - bz - a$ are subnormal if, and only if, $-3 < c' < 3$, $a + b + c' - 1 < 0$, $-a + b - c' - 1 < 0$ and $b < a^2 - c'a - 1$.

4 Special Cases

It is interesting to consider two special cases of (2):

$$u_n = \alpha u_{n-2} + \beta u_{n-3}, \ n \in N, \quad (14)$$

and

$$u_n = \alpha u_{n-1} + \beta u_{n-3}, \ n \in N. \quad (15)$$

Their characteristic polynomials are $z^3 - \alpha z - \beta$ and $z^3 - \alpha z^2 - \beta$ respectively. Thus (14) is asymptotically stable if, and only if, (α, β) lies in the plane region defined by

$$\alpha + \beta - 1 < 0, \ \alpha - \beta - 1 < 0, \ \alpha < \beta^2 - 1,$$

while (15) is asymptotically stable if, and only if, (α, β) lies in the plane region defined by

$$-3 < \alpha < 3, \ \alpha + \beta - 1 < 0, \ \beta - \alpha - 1 < 0, \ 0 < \beta^2 - \alpha\beta - 1.$$

References

[1] G. Gandolfo, Economic Dynamics: Methods and Models, North-Holland, 1980.

[2] E. J. Barbeau, Polynomials, Springer-Verlag, 1989.