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More On A Rational Recurrence Relation*

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Abstract

In this note we give some additional information on the behavior of the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}x_n}, \quad n = 0, 1...$$

where the initial conditions x_{-1}, x_0 are real numbers.

1 Introduction

In this note we consider the following nonlinear difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_{n-1}x_n}, \quad n = 0, 1... \,. \tag{1}$$

Recently there has been a great interest in studying the global attractivity, the boundedness character and the periodic nature of nonlinear difference equations. Rational and nonrational difference equations are systematically studied by the author of this note and his collaborators see, for example, [3-5,7-26] and the references therein. This note is motivated by the short note [1].

In [1], the author proved, by induction, that the following formula

$$x_n = \begin{cases} x_{-1}\dfrac{\prod_{i=0}^{[(n+1)/2]-1}(2x_{-1}x_0 i+1)}{\prod_{j=0}^{[(n+1)/2]-1}((2i+1)x_{-1}x_0+1)} & , \quad \text{for } n \text{ odd} \\ x_0\dfrac{\prod_{i=1}^{n/2}((2i-1)x_{-1}x_0+1)}{\prod_{i=1}^{n/2}(2ix_{-1}x_0+1)} & , \quad \text{for } n \text{ even} \end{cases} \tag{2}$$

holds for all positive solutions of Eq.(1). Nothing is mentioned about the global attractivity, the boundedness character and the periodic nature of the equation.

The purpose of this note is to explain what stands behind the "mysterious" explicit formula (2) and to give some additional information on properties of the solutions of Eq.(1).

The following closely related equation was considered by Stević in [20]

$$x_{n+1} = \frac{x_{n-1}}{g(x_n)}, \tag{3}$$

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where $g(x)$ is a continuous positive function on the interval $[0, \infty)$ such that $g(0) = 1$. Motivation for [20] stems from [2], where Ladas and his collaborators posed the following problem:

Is there a solution of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{1 + x_n}, \quad x_{-1}, x_0 > 0, \quad n = 0, 1, 2, ... \tag{4}$$

such that $x_n \to 0$ *as* $n \to \infty$?

Solution of this very difficult problem can be found in [20] where Stević obtained a result for more general equation (3). Here is the theorem.

THEOREM A. Let g be a C^1 increasing function defined on $\mathbf{R}_+$ such that $g(0) = 1$. Suppose (x_n) is a solution of Eq.(3) with x_{-1}, $x_0 > 0$ and $x_0 g(x_0) > x_{-1}$. Then this solution tends to zero as $n \to \infty$.

It is clear from (1) and (4) that nonnegative solutions of Eq.(1) and Eq.(4) satisfy the following property

$$x_{n+1} \leq x_{n-1}, \qquad n = 0, 1, 2, ...,$$

which implies that the sequences (x_{2n}) and (x_{2n+1}) have finite limits, say l and L. Letting $n \to \infty$ in (1) and (4) we easily obtain that $lL = 0$. Theorem A shows that there are positive solutions of Eq.(3) which tend to zero as $n \to \infty$. A natural question is: *Is there a solution of Eq.*(1) *such that* $x_n \to 0$ *as* $n \to \infty$?

It is clear that Eqs.(1) and (4) have a unique equilibrium $\bar{x} = 0$. Hence, if a solution of Eq.(1) or Eq.(4) converges, its limit is equal to zero. One can suspect that all nonnegative solutions converge to the equilibrium. But it is not true.

Indeed, notice that the 2-periodic sequence of the form

$$..., p, 0, p, 0, p, 0, ...$$

where $p \in \mathbf{R}$, is a solution of Eqs.(1) and (4). This implies that there are solutions of Eqs.(1) and (4) which do not converge.

2 Solvability of equation (1)

Formula (2) inspired us to find a reasonable answer in order to get the formula directly. If $x_0 = 0$ or $x_{-1} = 0$, then we can easily obtain that

$$x_{2n} = 0 \qquad \text{or} \qquad x_{2n-1} = 0, \quad \text{for} \quad n \geq 0,$$

and consequently

$$x_{2n+1} = x_{-1} \qquad \text{or} \qquad x_{2n} = x_0, \quad \text{for} \quad n \geq 0.$$

Thus, let us assume that x_{-1} and x_0 are positive. Then it is clear that $x_n > 0$ for all $n \geq -1$.

Multiplying (1) by x_n and using the transformation $y_n = x_{n+1} x_n$ we obtain

$$y_n = \frac{y_{n-1}}{1 + y_{n-1}}. \tag{5}$$

Since $y_n > 0, n \geq -1$, Eq.(5) can be written in the form

$$\frac{1}{y_n} = \frac{1}{y_{n-1}} + 1. \tag{6}$$

Form (6) we easily obtain

$$\frac{1}{y_n} = \frac{1}{y_{-1}} + n + 1$$

that is

$$x_{n+1}x_n = \left(n + 1 + \frac{1}{x_0 x_{-1}}\right)^{-1} = \left(\frac{a}{(n+1)a+1}\right), \tag{7}$$

where $a = x_0 x_{-1}$.

From (7) we obtain

$$x_{2n} = \frac{1}{x_{2n-1}}\left(\frac{a}{2na+1}\right), \tag{8}$$

and

$$x_{2n-1} = \frac{1}{x_{2n-2}}\left(\frac{a}{(2n-1)a+1}\right) \tag{9}$$

for $n \geq 0$. Using (8) and (9) we get

$$x_{2n} = x_{2n-2}\frac{(2n-1)a+1}{2na+1}$$

and

$$x_{2n+1} = x_{2n-1}\frac{2na+1}{(2n+1)a+1}$$

from which formula (2) it follows.

REMARK 1. Note that if $x_0 x_{-1} \neq -1/n$, for all $n \geq 1$, then formula (2) also represents solutions of Eq.(1) when x_0 and x_{-1} are real numbers.

3 Attractivity of solutions

Since we have an explicit formula for solutions of Eq.(1) we can use it in investigating their behavior. Let $a = x_0 x_{-1}$. The main result is the following:

THEOREM 1. Let $a \neq 0$ and $a \neq -1/n, n \in \mathbf{N}$. Then every solution of Eq.(1) converges to zero.

PROOF. Let (x_n) be an arbitrary solution of Eq.(1). It is enough to prove that the subsequences (x_{2n}) and (x_{2n-1}) converge to zero as $n \to \infty$. From (2) we have

$$\begin{aligned} |x_{2n}| &= |x_0|\frac{\prod_{i=1}^{n}((2i-1)a+1)}{\prod_{i=1}^{n}(2ia+1)} = |x_0|\exp\left(\sum_{i=1}^{n}\ln\ \frac{(2i-1)a+1}{2ia+1}\right) \\ &= |x_0|\exp\left(\sum_{i=1}^{n}\ln\ \left(1 - \frac{a}{2ia+1}\right)\right) \end{aligned}$$

$$= \quad |x_0|c(n_0)\exp\left(-a\sum_{i=n_0}^{n}\left(\frac{1}{2ia+1}+\mathcal{O}\left(\frac{1}{i^2}\right)\right)\right)\to 0, \quad \text{as} \quad n\to\infty$$

since $\sum_{i=1}^{n}\frac{1}{2ia+1}\to +\infty\,(\operatorname{sign} a)$ as $n\to\infty$ and since the series

$$\sum_{i=n_0}^{\infty}\mathcal{O}\left(\frac{1}{i^2}\right)$$

is convergent. Here $c(n_0)$ is a positive constant depending on $n_0\in\mathbf{N}$.

Similarly we obtain

$$\begin{aligned} |x_{2n+1}| &= |x_{-1}|\exp\left(\sum_{i=0}^{n}\ln\left(1-\frac{a}{(2i+1)a+1}\right)\right) \\ &= |x_{-1}|C(n_1)\exp\left(-a\sum_{i=n_1}^{n}\left(\frac{1}{(2i+1)a+1}+\mathcal{O}\left(\frac{1}{i^2}\right)\right)\right)\to 0, \quad \text{as} \quad n\to\infty, \end{aligned}$$

as desired.

COROLLARY 1. Every positive solution of Eq.(1) converges to zero.

The following result is already mentioned in the introduction.

THEOREM 2. Let $a=0$. Then every solution of Eq.(1) is 2-periodic (not necessarily prime).

REMARK 2. Note that the equation

$$x_{n+1}=\frac{bx_{n-1}}{b+cx_{n-1}x_n}, \quad n=0,1...$$

where b and c are positive real numbers, can be reduced to Eq.(1) by the change $x_n=\sqrt{b/c}\,y_n$.

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