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Characterization Of Polynomials Using Reflection Coefficients*

José Luis Díaz-Barrero[†], Juan José Egozcue[‡]

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Abstract

A characterization of polynomials by means of their reflection coefficients is presented. A complete classification of the set of all polynomials is obtained and two theorems on self-inversive polynomials are given.

1 Introduction

Polynomials can be defined in terms of their coefficients and/or their zeros. They can also be characterized by their reflection coefficients using Schur-Cohn type recursions ([9], [10]). Polynomials characterized by reflection coefficients have been used efficiently in many applications in control theory, signal processing and system identification areas ([1], [3], [8], [12]). Although reflection coefficients and polynomials appear frequently in studies of polynomials that are orthogonal on the unit circle ([6], [13]), they have never been explicitly characterized and classified. The aim of this paper is to give a complete characterization and classification of polynomials using reflection coefficients instead of zeros and coefficients.

In order to define our characterization, we first introduce some notations and known results. Let $A_n(z)$ be a monic complex polynomial of degree n, namely,

$$A_n(z) = z^n + a_{n,n-1}z^{n-1} + \cdots + a_{n1}z + a_{n0}. \tag{1}$$

The reciprocal polynomial $A_n^*(z)$ of $A_n(z)$ is defined by

$$A_n^*(z) = z^n\overline{A_n(1/\overline{z})} = \sum_{k=0}^{n} \overline{a}_{n,n-k}z^k \tag{2}$$

Here $\overline{a}$ denotes the complex conjugation of a. If there exists an unitary complex number u, such that $A_n(z) = uA_n^*(z)$, we say $A_n(z)$ is self-inversive (or self–reciprocal). Notice that its zeros lie on or are symmetric in the unit circle.

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†Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, 08034 Barcelona, Spain

‡Applied Mathematics III, Universitat Politècnica de Catalunya, Jordi Girona 1-3, C2, 08034 Barcelona, Spain

The reflection coefficients $\alpha_k's$, also known in the literature as Schur-Szegö parameters [5] or partial correlation (PARCOR) coefficients [8], can be obtained from $A_n(z)$ by using backward Levinson's recursion (implicitly given by Levinson in [9]; see also [12])

$$zA_{k-1}(z) = \frac{1}{1-|\alpha_k|^2}\left[A_k(z) - \alpha_k A_k^*(z)\right] \tag{3}$$

where $\alpha_k = a_{k0}$. From (3), and some straightforward algebra, the forward recursion

$$A_k(z) = zA_{k-1}(z) + \alpha_k A_{k-1}^*(z), \tag{4}$$

is obtained.

REMARK. From (3) and (4) the coefficient expressions of $A_{k-1}(z)$ and $A_k(z)$ are given respectively by

$$A_{k-1}(z) = \frac{1}{1-|\alpha_k|^2}\left[\sum_{j=0}^{k-1}\left(a_{k,j+1} - \alpha_k \overline{a}_{k,k-1-j}\right) z^j\right]$$

and

$$A_k(z) = \sum_{j=0}^{k}\left(a_{k-1,j-1} + \alpha_k \overline{a}_{k-1,k-1-j}\right) z^j.$$

Note that in the last expressions we have considered all the coefficients with negative subscript equal to zero.

We close this section stating a key and well known result in the theory of polynomials and reflection coefficients ([1], [3], [11], [14]) that we will use further on.

THEOREM 1. Let $A_n(z) = \sum_{k=0}^{n} a_{nk} z^k$ be a monic complex polynomial with reflection coefficients $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then, $A_n(z)$ has all its zeros inside the unit disk if, and only if, $|\alpha_k| < 1$ for $k = 1, 2, \ldots, n$.

2 Characterization of Polynomials

Now we can define the characterization of $A_n(z)$ by its reflection coefficients. It can be stated as follows.

DEFINITION 1. The characterization of the monic complex polynomial $A_n(z)$ using reflection coefficient is given by

$$A_n(z) \equiv \left[A_j(z); \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_n\right]_r, \tag{5}$$

where $0 \leq j \leq n$ and $A_j(z)$, called base polynomial, is either $A_j(z) = A_0(z) = 1$ for $j = 0$ or $A_j(z)$ is a non-self-inversive unitary polynomial (i.e., $|a_{j0}| = 1$) for $1 \leq j \leq n$. The $\alpha_k \in C,\ k = j+1, j+2, \ldots, n$, are reflection coefficients.

We remark that in (5), the subscript r means reflection coefficient characterization. Frequently $A_j(z) = A_0(z) = 1$; then the reference to polynomial $A_j(z)$ may be omitted.

The preceding recursion (3) cannot be carried out whenever an unitary reflection coefficient is found (i.e., $|\alpha_k| = 1$). In that case, recursion (3) is stopped. However, backward recursion can be carried out if the polynomial $A_k(z)$ is self-inversive, as we will see later on. But, if $A_k(z)$ is not self-inversive, backward recursion is impossible and the characterization (5) holds with $[A_k(z); \alpha_{k+1}, \ldots, \alpha_n]_r$.

If $|\alpha_k| \neq 1$ for $k = 1, 2, \ldots, n$, then $A_j(z)$ in (5) becomes $A_0(z) = 1$ and (5) is a pure reflection coefficient characterization.

From (3), every polynomial of degree n can be characterized by $[1; \alpha_1, \alpha_2, \ldots, \alpha_n]_r$, with base polynomial $A_0(z) = 1$ if, and only if, its reflection coefficients are not unitary.

THEOREM 2. If a polynomial $A_n(z)$ can be characterized by non-unitary reflection coefficients with base polynomial $A_0(z) = 1$, then the characterization is unique.

PROOF. Assume that we have two characterizations for $A_n(z)$. That is,

$$A_n(z) \equiv [1; \alpha_1, \alpha_2, \ldots, \alpha_n]_r \equiv [1; \beta_1, \beta_2, \ldots, \beta_n]_r .$$

We have to prove that $\alpha_k = \beta_k$ for all $k = 1, 2, \ldots, n$. We start with $A_0(z) = B_0(z) = A_0^*(z) = B_0^*(z) = 1$. By applying forward recursion (4), we generate the polynomials

$$\begin{array}{cc} A_1(z) & B_1(z) \\ A_2(z) & B_2(z) \\ \cdots & \cdots \\ A_n(z) & B_n(z). \end{array}$$

Then, from $A_n(z) = B_n(z)$, we get $\alpha_n = a_{n0} = b_{n0} = \beta_n$. Now, by using backward recursion, we have (observe that the application of (3) requires non-unitary reflection coefficients)

$$zA_{n-1}(z) = \frac{1}{1 - |\alpha_n|^2} \left[A_n(z) - \alpha_n A_n^*(z) \right] = zB_{n-1}(z).$$

Hence, $\alpha_{n-1} = a_{n-1,0} = b_{n-1,0} = \beta_{n-1}$.

This procedure can be applied repeatedly and this implies $\alpha_k = \beta_k$, for all $k = 1, 2, \ldots, n$, and we are done.

As far as we know, Theorem 2 has been partially treated, specially for $|\alpha_k| < 1$ (e.g. [3]). But the case $|\alpha_k| > 1$ has been studied only in a weak form.

After establishing the uniqueness of the pure reflection characterization, a more general question arises: what happens when an unitary reflection coefficient appears before applying backward recursion (3)? This question is addressed in the literature only partially, when backward recursion is carried out by using the derivative of the polynomial. This has appeared in handling the classical zero counter problems ([2],[4],[7]).

To answer the preceding question we state and prove the main result of this paper.

THEOREM 3. Every polynomial can be characterized by using reflection coefficients. The characterization is unique if $A_n(z)$ and all the polynomials that can be obtained from it by applying (3) recursively are not self-inversive. If by applying (3) recursively to $A_n(z)$, a self-inversive polynomial is obtained, then $A_n(z)$ can be characterized by reflection coefficients, but this characterization is not unique.

PROOF. If no unitary reflection coefficients appear when applying backward recursion (3), the statement clearly follows from Theorem 2. Otherwise, assume that after some iterations of (3) (eventually the first), a polynomial $A_k(z)$ with such reflection coefficient α_k ($|\alpha_k| = 1$) is obtained. Then, recursion (3) stops and the process can not be iterated again. Now, we consider the generic monic polynomial $A_{k-1}(z) = \sum_{j=0}^{k-1} a_{k-1,j} z^j$, and we apply forward recursion (4) to it with α_k. We get $A_k(z) = zA_{k-1}(z) + \alpha_k A^*_{k-1}(z)$ which, in coefficients notation, reads

$$\sum_{j=0}^{k} a_{kj} z^j = \sum_{j=0}^{k} \left(a_{k-1,j-1} + \alpha_k \overline{a}_{k-1,k-1-j} \right) z^j \tag{6}$$

Identifying coefficients of equal powers in (6), and carrying out some straightforward algebra, the system of linear equations

$$\alpha_k \overline{a}_{k-1,j-1} + a_{k-1,k-1-j} = \alpha_k \overline{a}_{kj}, \; j = 1, 2, \ldots, k-1 \tag{7}$$

follows. Applying Rouché-Frobenius theorem to (7), we have:

(a) The polynomial is self-inversive, namely $|a_{kj}| = |a_{k,k-j}|$, for $j = 0, 1, \ldots, k$. In this case, system (7) is compatible and has infinitely many solutions. This tells us that there are an infinite number of ways of going from $A_{k-1}(z)$ to $A_k(z)$.

(b) The polynomial is not self-inversive. That is, there exist at least one subscript $j \in \{0, 1, 2, \ldots, k\}$, such that $|a_{kj}| \neq |a_{k,k-j}|$. Then, system (7) does not have any solution. Consequently, the characterization of $A_n(z)$ is unique and it can be written as $A_n(z) \equiv [A_k(z); \alpha_{k+1}, \ldots, \alpha_n]_r$.

This completes the proof of Theorem 3.

We close this section by giving a classification of the set $C[z]$ of all monic complex polynomials of any degree. We call *canonical representative polynomial*, namely $A_j(z)$, to every unitary polynomial in $C[z]$ that is not self-inversive and define

$$\mathcal{C}(A_j) = \left\{ A_k(z) = [A_j(z); \alpha_{j+1}, \ldots, \alpha_k]_r \, , |\alpha_s| \neq 1, j+1 \leq s \leq k \right\} \tag{8}$$

with $j \leq k$ and $\alpha_s \in C$. Next, we also define the class $\mathcal{C}(1)$ as the class that contains all the polynomials that can be obtained from $A_0(z) = 1$ and whatever reflection coefficients. That is,

$$\mathcal{C}(1) = \{ A_k(z) = [1; \alpha_1, \alpha_2, \ldots, \alpha_k]_r \, , \, 1 \leq k \} \tag{9}$$

Finally, we define

$$A_j(z) \sim B_k(z) \Longleftrightarrow \mathcal{C}(A_j) \equiv \mathcal{C}(B_k) \tag{10}$$

From Theorem 3, every polynomial belongs to only one class. The preceding classes $\mathcal{C}(A_j)$ contain their canonical representatives. These canonical representatives are all unitary and non-self-inversive polynomials. Furthermore, all self-inversive polynomials belongs to $\mathcal{C}(1)$ although they may be obtained from an unitary non-self-inversive polynomial by using unitary reflection coefficients.

From the definition of an equivalence class, two classes do not overlap and they are a partition of $C[z]$. Hence, (10) is an equivalence relation that produces a classification in the set of complex polynomials. After this classification, we point out that in the literature concerning reflection coefficients, the polynomials usually considered belong to $\mathcal{C}(1)$ class, but they are often restricted to those which reflection coefficients are of modulus less than 1.

The above results show that self–inversive polynomials and polynomials whose reflection coefficients are of modulus greater than 1 share with the preceding polynomials some properties. For instance, they can be obtained from $A_0(z) = 1$ by forward Levinson recursion (4).

3 Self-inversive Polynomials and Reflection Coefficients

The preceding statement announces that a self-inversive polynomial $A_n(z)$ can be obtained from polynomials of degree $n-1$. One way of doing so is to obtain it from its derivative ([4], [7]). This known result, which is useful when handling with classical zero counters, can be stated as follows

THEOREM 4. Every self–inversive polynomial can be obtained from its derivative, normalized to monic, by applying forward recursion (4) with an appropriate reflection coefficient.

In what follows, we exhibit another explicit way to do it. This procedure uses reflection coefficients as explicit functions of the zeros of $A_n(z)$.

THEOREM 5. Let $A_n(z) = (z - z_r)A_{n-1}(z)$ be a self-inversive polynomial and let z_r be a zero of $A_n(z)$ on the unit circle. Then $A_n(z) = zA_{n-1}(z) + \alpha_n A^*_{n-1}(z)$ with $\alpha_n = -z_r \alpha_{n-1}$.

PROOF. Since the zeros of a self-inverse polynomial lie on the unit circle $|z| = 1$ or are symmetric in the unit circle, we assume that $A_n(z)$ can be factored into

$$A_n(z) = \prod_{j=1}^{r}(z - z_j) \prod_{k=1}^{s}(z - z_k)\left(z - \frac{1}{\overline{z}_k}\right) \tag{11}$$

where $|z_j| = 1$ for $j = 1, 2, \ldots, r$, $|z_k| < 1$ for $k = 1, 2, \ldots, s$ and $r + 2s = n$.

Setting $A_{n-1}(z) = \prod_{j=1}^{r-1}(z - z_j) \prod_{k=1}^{s}(z - z_k)\left(z - \frac{1}{\overline{z}_k}\right)$, we have

$$A^*_{n-1}(z) = (-1)^{n-1} \prod_{j=1}^{n-1} \overline{z}_j \prod_{j=1}^{r-1}(z - z_j) \prod_{k=1}^{s}(z - z_k)\left(z - \frac{1}{\overline{z}_k}\right) = \overline{\alpha}_{n-1} A_{n-1}(z)$$

and

$$zA_{n-1}(z) + \alpha_n A^*_{n-1}(z) = zA_{n-1}(z) + \alpha_n \overline{\alpha}_{n-1} A_{n-1}(z) = (z + \alpha_n \overline{\alpha}_{n-1})A_{n-1}(z) \tag{12}$$

By equating (11) and (12), it follows

$$(z - z_r)A_{n-1}(z) = (z + \alpha_n \overline{\alpha}_{n-1})A_{n-1}(z) \tag{13}$$

Since α_{n-1} is unitary, from (13), we get $\alpha_n = -z_r\alpha_{n-1}$ and we are done.

THEOREM 6. Let $A_n(z) = (z-z_s)(z-1/\overline{z}_s)A_{n-2}(z)$ be a self-inversive polynomial. Then $A_n(z)$ can be obtained from $A_{n-2}(z)$ by applying forward recursion (4) two times with reflection coefficients $\alpha_{n-1} = -\frac{1}{2}e^{i\theta_s}\left(r_s + \frac{1}{r_s}\right)\alpha_{n-2}$ and $\alpha_n = e^{i2\theta_s}\alpha_{n-2}$, where $z_s = r_s e^{i\theta_s}$.

PROOF. Assume that $A_n(z)$ can be factored into

$$A_n(z) = \prod_{j=1}^{r}(z-z_j)\prod_{k=1}^{s}(z-z_k)\left(z - \frac{1}{\overline{z}_k}\right)$$

where $|z_j| = 1$, $|z_k| < 1$, $k = 1, 2, \ldots, s$ and $r + 2s = n$. We will prove that there exist two reflection coefficients α_{n-1} and α_n such that

$$A_n(z) = A_{n-2}(z)B_2(z),$$

where

$$A_{n-2}(z) = \prod_{j=1}^{r}(z-z_j)\prod_{k=1}^{s-1}(z-z_k)\left(z - \frac{1}{\overline{z}_k}\right)$$

and

$$B_2(z) = (z-z_s)\left(z - \frac{1}{\overline{z}_s}\right) = (z - r_s e^{i\theta_s})\left(z - \frac{1}{r_s}e^{i\theta_s}\right) = z^2 - e^{i\theta_s}\left(r_s + \frac{1}{r_s}\right)z + e^{i2\theta_s};$$

can be obtained from $A_{n-2}(z)$ by applying forward recursion (4) with reflection coefficients α_{n-1} and α_n. To carry out the first iteration we set $\alpha_{n-1} = -\frac{1}{2}e^{i\theta_s}\left(r_s + \frac{1}{r_s}\right)\alpha_{n-2}$ and we get

$$A_{n-1}(z) = zA_{n-2}(z) - \frac{1}{2}e^{i\theta_s}\left(r_s + \frac{1}{r_s}\right)\alpha_{n-2}A^*_{n-2}(z) \tag{14}$$

Since

$$A^*_{n-2}(z) = \overline{\alpha}_{n-2}\prod_{j=1}^{r}(z-z_j)\prod_{k=1}^{s-1}(z-z_k)\left(z - \frac{1}{\overline{z}_k}\right) = \overline{\alpha}_{n-2}A_{n-2}(z),$$

then (14) can be written as

$$\begin{aligned} A_{n-1}(z) &= zA_{n-2}(z) - \frac{1}{2}e^{i\theta_s}(r_s + \frac{1}{r_s})\alpha_{n-2}\overline{\alpha}_{n-2}A_{n-2}(z) \\ &= A_{n-2}(z)\left\{z - \frac{1}{2}e^{i\theta_s}\left(r_s + \frac{1}{r_s}\right)\right\}. \end{aligned}$$

Before running forward recursion (4) again, we observe that

$$\begin{aligned} A^*_{n-1}(z) &= z^{n-1}\overline{A_{n-2}(1/\overline{z})}\left\{\frac{1}{z} - \frac{1}{2}e^{-i\theta_s}\left(r_s + \frac{1}{r_s}\right)\right\} \\ &= z^{n-2}\overline{A_{n-2}(1/\overline{z})}\left\{1 - \frac{1}{2}e^{-i\theta_s}\left(r_s + \frac{1}{r_s}\right)z\right\} \\ &= \overline{\alpha}_{n-2}A_{n-2}(z)\left\{1 - \frac{1}{2}e^{-i\theta_s}\left(r_s + \frac{1}{r_s}\right)z\right\}. \end{aligned}$$

Next, we take $\alpha_n = e^{i2\theta_s}\alpha_{n-2}$ and apply forward recursion (4) again, we get

$$\begin{aligned} A_n(z) &= zA_{n-2}(z)\{z - \frac{1}{2}e^{i\theta_s}\left(r_s + \frac{1}{r_s}\right)\} \\ &+ e^{i2\theta_s}\alpha_{n-2}\overline{\alpha}_{n-2}A_{n-2}(z)\left\{1 - \frac{1}{2}e^{-i\theta_s}\left(r_s + \frac{1}{r_s}\right)z\right\} \\ &= A_{n-2}(z)\left\{z^2 - e^{i\theta_s}\left(r_s + \frac{1}{r_s}\right)z + e^{i2\theta_s}\right\} = A_{n-2}(z)B_2(z). \end{aligned}$$

This completes the proof of Theorem 6.

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