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Multi-Point Boundary Value Problems For Higher Order Differential Equations*

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Abstract

Using the Leggett-Williams fixed point theorem, we establish existence results for solutions to m-point boundary value problem for a $2n$-th order differential equation under multipoint boundary conditions. In order to obtain our results, the associated Green's function for the above problem is also given.

1 Introduction

In this paper we shall consider the $2n$-th order m-point boundary value problem

$$\begin{cases} y^{(2n)}(t) = f(t, y(t), y''(t), ..., y^{(2(n-1))}(t)), \quad 0 \le t \le 1, \\ y^{(2i+1)}(0) = 0, \quad y^{(2i)}(1) = \sum_{j=1}^{m-2} k_{ij} y^{(2i)}(\xi_j), \quad 0 \le i \le n-1, \end{cases} \tag{1}$$

where $(-1)^n f : [0,1] \times R^n \to [0,\infty)$ is continuous, and $k_{ij} > 0$ for $i = 0, 1, ..., n-1$ and $j = 1, 2, ..., m-2$, and $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \xi_{m-1} = 1$. The following conditions will be assumed throughout: (A_1) $k_{ij} > 0$ for $i = 0, 1, ..., n-1$ and $j = 1, 2, ..., m-2$, $0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < \xi_{m-1} = 1$ and $1 - \sum_{j=1}^{m-2} k_{ij} > 0$; (A_2) $(-1)^n f : [0,1] \times R^n \to [0,\infty)$ is continuous.

In recent years, there is much attention focused on questions of positive solutions of multiple-point boundary value problems for ordinary differential equations [1-5]. Much of this interest is due to the applicability of certain fixed point theorems of Krasnoselskii or Leggett and Williams to obtain positive solutions or multiple positive solutions which lie in a cone.

The multi-point boundary value problems for ordinary order differential equations arise in a variety of different areas of applied mathematics and physics. In [6], Il'in and Moiseev first studied multi-point boundary value problems for linear second order ordinary differential equations. Since then, many authors [7-8] have also discussed nonlinear second order multi-point boundary value problems. Recently, Ma [3] used

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Krasnoselskii's fixed point theorem in cones [9] to prove the existence of positive solutions for the three-point boundary value problem

$$\begin{cases} u'' + a(t)g(u) = 0, \quad 0 < t < 1, \\ u(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases} \tag{2}$$

where $\alpha > 0$, $\eta \in (0,1)$, $\alpha\eta < 1$, $a \in C([0,1],[0,\infty))$, and $g \in C([0,\infty),[0,\infty))$ is either superlinear or sublinear.

For the $2n$-th order two point boundary value problem

$$\begin{cases} y^{(2n)} = f(y(t), y''(t), ..., y^{(2(n-1))}(t)), \quad 0 \le t \le 1, \\ \qquad y^{(2i)}(0) = y^{(2i)}(1) = 0, \quad 0 \le i \le n-1, \end{cases} \tag{3}$$

Davis et al. [10] imposed growth conditions on f to yield at least three symmetric positive solutions to (3) by applying the Leggett-Williams fixed point theorem [11]. Motivated by the above results, in this paper we study the existence of multiple positive solutions for the problem (1). In order to obtain our result, we first give the associated Green's function for the problem (1), which is the base for further discussion. Using the Leggett-Williams fixed point theorem and the Green's function, we get that the boundary value problem (1) has at least three positive solutions.

2 Main Results

We begin with some known results.

DEFINITION 1. Suppose K is a cone in a Banach. The map α is a nonnegative continuous concave functional on K provided $\alpha : K \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \ge t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in K$ and $0 \le t \le 1$.

DEFINITION 2. Let $0 < a < b$ be given and let α be a nonnegative continuous concave functional on K. Define the convex sets P_r and $P(\alpha, a, b)$ by

$$P_r = \{x \in K \mid ||x|| < r\}$$

and

$$P(\alpha, a, b) = \{x \in K \mid a \le \alpha(x), ||x|| \le b\}.$$

THEOREM 1 (Leggett-Williams Fixed Point Theorem [11]). Let $A : \overline{P_c} \to \overline{P_c}$ be a completely continuous operator and let α be a nonnegative continuous concave functional on K such that $\alpha(x) \le ||x||$ for all $x \in \overline{P_c}$. Suppose there exist $0 < a < b < d \le c$ such that

(C_1) $\{x \in P(\alpha, b, d) | \alpha(x) > b\} \ne \emptyset$ and $\alpha(Ax) > b$ for $x \in P(\alpha, b, d)$,

(C_2) $||Ax|| < a$ for $||x|| \le a$, and

(C_3) $\alpha(Ax) > b$ for $x \in P(\alpha, b, c)$ with $||Ax|| > d$.

Then A has at least three fixed points x_1, x_2, and x_3 such that

$$||x_1|| < a, \quad b < \alpha(x_2), \quad \text{and} \quad ||x_3|| > a \ \text{ with } \ \alpha(x_3) < b.$$

LEMMA 1. Suppose $\sum_{i=1}^{m-2} k_i \neq 1$. If $y(t) \in C[0,1]$, then the problem

$$\begin{cases} u''(t) + y(t) = 0, \quad 0 \le t \le 1, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i) \end{cases} \tag{4}$$

has a unique solution

$$\begin{aligned} u(t) &= -\int_0^t (t-s)y(s)ds + \frac{1}{1 - \sum_{i=1}^{m-2} k_i} \int_0^1 (1-s)y(s)ds \\ &\quad - \frac{1}{1 - \sum_{i=1}^{m-2} k_i} \sum_{i=1}^{m-2} k_i \int_0^{\xi_i} (\xi_i - s)y(s)ds. \end{aligned}$$

The proof follows from direct verification.

LEMMA 2. Suppose $0 < \sum_{i=1}^{m-2} k_i < 1$. If $y \in C[0,1]$ and $y \ge 0$, then the unique solution u of (4) satisfies

$$\inf_{t \in [\xi_1, 1]} u(t) \ge \gamma ||u||,$$

where $\gamma = \sum_{i=1}^{m-2} k_i(1-\xi_i)/(1 - \sum_{i=1}^{m-2} k_i \xi_i)$, and $||u||$ is the maximum of u on $[0,1]$.

PROOF. Obviously, $u(t)$ is maximum at $t = 0$, i.e., $||u|| = u(0)$. The concavity of u implies

$$\frac{u(\xi_i) - u(1)}{1 - \xi_i} \ge u(0) - u(1), \ 1 \le i \le m-2.$$

So,

$$u(\xi_i) - u(1) \ge (u(0) - u(1))(1 - \xi_i),$$

i.e.,

$$u(\xi_i) - \xi_i u(1) \ge u(0)(1 - \xi_i), \ 1 \le i \le m-2.$$

Therefore,

$$\sum_{i=1}^{m-2} k_i(u(\xi_i) - \xi_i u(1)) \ge \sum_{i=1}^{m-2} k_i u(0)(1 - \xi_i).$$

From (4), we have

$$u(1) \ge \frac{\sum_{i=1}^{m-2} k_i(1-\xi_i)}{1 - \sum_{i=1}^{m-2} k_i \xi_i} u(0).$$

Thus, $\inf_{t \in [\xi_1, 1]} u(t) \ge \gamma ||u||$.

LEMMA 3. Suppose $0 < \sum_{i=1}^{m-2} k_i < 1$. The Green's function for the boundary value problem

$$\begin{cases} -u''(t) = 0, \quad 0 \le t \le 1, \\ u'(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i) \end{cases}$$

is given by

$$G(t,s) = \frac{(1-t) - \sum_{i=1}^{m-2} k_i(\xi_i - t)}{1 - \sum_{i=1}^{m-2} k_i}$$

for $0 \le t \le 1, 0 \le s \le \xi_1, s \le t$;

$$G(t,s) = \frac{(1-t) - \sum_{j=i}^{m-2} k_j(\xi_j - t) + \sum_{j=1}^{i-1} k_j(t-s)}{1 - \sum_{i=1}^{m-2} k_i}$$

for $\xi_{r-1} \le t \le \xi_r, 2 \le r \le m-1, \xi_{i-1} \le s \le \xi_i, 2 \le i \le r, s \le t$;

$$G(t,s) = \frac{(1-s) - \sum_{j=i}^{m-2} k_j(\xi_j - s)}{1 - \sum_{i=1}^{m-2} k_i}$$

for $\xi_{r-1} \le t \le \xi_r, 1 \le r \le m-2, \xi_{i-1} \le s \le \xi_i, r \le i \le m-2, t \le s$; and

$$G(t,s) = \frac{1-s}{1 - \sum_{i=1}^{m-2} k_i}$$

for $0 \le t \le 1, \xi_{m-2} \le s \le 1, t \le s$.

PROOF. For $0 \le t \le \xi_1$, the unique solution of (4) can be expressed as

$$\begin{aligned} u(t) \quad = \quad & \int_0^t \frac{(1-t) - \sum_{i=1}^{m-2} k_i(\xi_i - t)}{1 - \sum_{i=1}^{m-2} k_i} y(s)ds + \int_t^{\xi_1} \frac{(1-s) - \sum_{i=1}^{m-2} k_i(\xi_i - s)}{1 - \sum_{i=1}^{m-2} k_i} y(s)ds \\ & + \sum_{i=2}^{m-2} \int_{\xi_{i-1}}^{\xi_i} \frac{(1-s) - \sum_{j=i}^{m-2} k_j(\xi_j - s)}{1 - \sum_{i=1}^{m-2} k_i} y(s)ds + \int_{\xi_{m-2}}^1 \frac{1-s}{1 - \sum_{i=1}^{m-2} k_i} y(s)ds. \end{aligned}$$

For $\xi_{r-1} \le t \le \xi_r$, $2 \le r \le m-2$ and $\xi_{m-2} \le t \le 1$, we have similar expressions. Therefore, the unique solution of (4) is $u(t) = \int_0^1 G(t,s)y(s)ds$. Lemma 3 is now proved.

LEMMA 4. Suppose (A_1) holds. Then $g_i(t,s) \le 0$ for $0 \le i \le n-1$, where $g_i(t,s)$ is the Green's function for the problem

$$\begin{cases} u''(t) = 0, \quad 0 \le t \le 1, \\ u'(0) = 0, \quad u(1) = \sum_{j=1}^{m-2} k_{ij} u(\xi_j). \end{cases}$$

The proof follows from Lemma 3.

Let $G_1(t,s) = g_{n-2}(t,s)$. For $2 \le j \le n-1$, we define

$$G_j(t,s) = \int_0^1 g_{n-j-1}(t,r) G_{j-1}(r,s) dr.$$

LEMMA 5. Suppose (A_1) holds. If $y \in C[0,1]$, then the boundary value problem

$$\begin{cases} u^{(2l)}(t) = y(t), & 0 \le t \le 1, \\ u^{(2i+1)}(0) = 0, u^{(2i)}(1) = \sum_{j=1}^{m-2} k_{n-l+i-1,j} u^{(2i)}(\xi_j), & 0 \le i \le l-1 \end{cases} \tag{5}$$

has a unique solution for each $1 \le l \le n-1$, where $G_l(t,s)$ is the associated Green's function for the boundary value problem (5).

From Lemma 3, it is easy to see that the result holds by using induction.

For each $1 \le l \le n-1$, we define $A_l : C[0,1] \to C[0,1]$ by

$$A_l v(t) = \int_0^1 G_l(t,\tau) v(\tau) d\tau.$$

With the aid of Lemma 5, for each $1 \le l \le n-1$, we have

$$\begin{cases} (A_l v)^{(2l)}(t) = v(t), & 0 \le t \le 1, \\ (A_l v)^{(2i+1)}(0) = 0, (A_l v)^{(2i)}(1) = \sum_{j=1}^{m-2} k_{n-l+i-1,j} (A_l v)^{(2i)}(\xi_j), & 0 \le i \le l-1. \end{cases}$$

Therefore (1) has a solution if and only if the boundary value problem

$$\begin{cases} v''(t) = f(t, A_{n-1}v(t), A_{n-2}v(t), ..., A_1 v(t), v(t)), \ 0 \le t \le 1, \\ v'(0) = 0, \quad v(1) = \sum_{j=1}^{m-2} k_{n-1,j} v(\xi_j) \end{cases} \tag{6}$$

has a solution. If y is a solution to (1), then $v = y^{(2(n-1))}$ is a solution to (6). Conversely, if v is a solution to (6), then $y = A_{n-1}v$ is a solution to (1). In addition if $(-1)^{n-1} v(t) \ge 0 (\not\equiv 0)$ on $[0,1]$, then $y = A_{n-1}v$ is a positive solution to (1).

For $1 \le i \le n-1$, let $m_i = \min_{t \in [\xi_1, 1]} \int_{\xi_1}^1 |g_i(t,s)| ds$, and $M_i = \max_{t \in [0,1]} \int_0^1 |g_i(t,s)| ds$. Obviously, $0 < m_i < M_i$. Let $E = C[0,1]$ and define the cone $K \subset E$ to be the set of $u \in E$ such that $(-1)^{n-1} u$ is concave, nonnegative, nonincreasing on $[0,1]$ and $\min_{t \in [\xi_1, 1]} (-1)^{n-1} u(t) \ge \gamma ||u||$.

Finally, we define the nonnegative continuous concave functional α on K by

$$\alpha(u) = \min_{t\in[\xi_1,1]} |u(t)|.$$

For each $u \in K$, it is easy to see that $\alpha(u) \leq ||u||$.

THEOREM 2. Suppose (A_1) and (A_2) hold. In addition assume there exist nonnegative numbers a, b and c such that $0 < a < b \leq \min\{\gamma, \frac{m_{n-1}}{M_{n-1}}\}c$ and $f(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0)$ satisfies the following growth conditions:

(A_3) $f(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0) \leq \frac{c}{M_{n-1}}$ for $(t, |u_{n-1}|, |u_{n-2}|, \cdots, |u_0|) \in [0,1] \times \prod_{j=n-1}^{1} [0, \prod_{i=2}^{j+1} M_{n-i}c] \times [0, c]$;

(A_4) $f(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0) < \frac{a}{M_{n-1}}$ for $(t, |u_{n-1}|, |u_{n-2}|, \cdots, |u_0|) \in [0,1] \times \prod_{j=n-1}^{1} [0, \prod_{i=2}^{j+1} M_{n-i}a] \times [0, a]$;

(A_5) $f(t, u_{n-1}, u_{n-2}, \cdots, u_1, u_0) \geq \frac{b}{m_{n-1}}$ for $(t, |u_{n-1}|, |u_{n-2}|, \cdots, |u_0|) \in [\xi_1, 1] \times \prod_{j=n-1}^{1} [\prod_{i=2}^{j+1} m_{n-i}b, \prod_{i=2}^{j+1} M_{n-i}\frac{b}{\gamma}] \times [b, \frac{b}{\gamma}]$.

Then the boundary value problem (1) has at least three positive solutions u_1, u_2, u_3 such that

$$\left\| u_1^{(2(n-1))} \right\| < a, \quad b < \alpha\left(u_2^{(2(n-1))}\right)$$

and

$$\left\| u_3^{(2(n-1))} \right\| > a, \text{ with } \alpha\left(u_3^{(2(n-1))}\right) < b.$$

PROOF. We define the completely continuous operator A by

$$Au(t) = \int_0^1 g_{n-1}(t,s) f(s, A_{n-1}u(s), A_{n-2}u(s), ..., A_1u(s), u(s))ds.$$

If $u \in K$, with the use of Lemma 4, then $(-1)^{n-1}Au(t) \geq 0$. From the properties of $g_{n-1}(t,s)$, $((-1)^{n-1}Au)'(0) = 0$ and $((-1)^{n-1}Au)''(t) = (-1)^{n-1}f(t, A_{n-1}u(t), A_{n-2}u(t), \cdots, A_1u(t), u(t)) \leq 0$, $0 \leq t \leq 1$, so, $(-1)^{n-1}u$ is concave, nonnegative, nonincreasing on $[0,1]$. Using Lemma 2, $\min_{t\in[\xi_1,1]} (-1)^{n-1}Au(t) \geq \gamma||Au||$. Consequently, $A: K \to K$. If $u \in \overline{P_c}$, then $||u|| \leq c$. For $1 \leq j \leq n-1$,

$$||A_j u|| = \max_{t\in[0,1]} \left| \int_0^1 G_j(t,s)u(s)ds \right| \leq \prod_{i=2}^{j+1} M_{n-i}c.$$

From condition (A_3), we have

$$\begin{aligned} ||Au|| &= \max_{t\in[0,1]} |Au(t)| \\ &= \max_{t\in[0,1]} \left| \int_0^1 g_{n-1}(t,s) f(s, A_{n-1}u(s), A_{n-2}u(s), \cdots, A_1u(s), u(s))ds \right| \\ &\leq \frac{c}{M_{n-1}} \max_{t\in[0,1]} \int_0^1 |g_{n-1}(t,s)|ds = c. \end{aligned}$$

Therefore, $A : \overline{P_c} \to \overline{P_c}$.

Similarly, condition (C_2) of Theorem 1 holds by using (A_4).

We now show that condition (C_1) is satisfied. Clearly,

$$\left\{ u \in P(\alpha, b, \frac{b}{\gamma}) \middle| \alpha(u) > b \right\} \neq \emptyset.$$

If $u \in P(\alpha, b, \frac{b}{\gamma})$, then $b \le |u(t)| \le \frac{b}{\gamma}$ for $t \in [\xi_1, 1]$. For $\xi_1 \le t \le 1, \quad 1 \le j \le n-1$, we have

$$|A_j u(t)| \le \prod_{i=2}^{j+1} M_{n-i} \frac{b}{\gamma},$$

$$|A_j u(t)| = \left| \int_0^1 G_j(t,s) u(s) ds \right| \ge b \int_{\xi_1}^1 |G_j(t,s)| ds \ge \prod_{i=2}^{j+1} m_{n-i} b.$$

From condition (A_5), we get

$$\begin{aligned} \alpha(Au) &= \min_{t \in [\xi_1, 1]} \left| \int_0^1 g_{n-1}(t,s) f(s, A_{n-1}u(s), A_{n-2}u(s), \cdots, A_1 u(s), u(s)) ds \right| \\ &\ge \min_{t \in [\xi_1, 1]} \left| \int_{\xi_1}^1 g_{n-1}(t,s) f(s, A_{n-1}u(s), A_{n-2}u(s), \cdots, A_1 u(s), u(s)) ds \right| \\ &> \frac{b}{m_{n-1}} \min_{t \in [\xi_1, 1]} \int_{\xi_1}^1 |g_{n-1}(t,s)| ds = b. \end{aligned}$$

Therefore, condition (C_1) is satisfied.

Finally, we show that condition (C_3) holds. If $u \in P(\alpha, b, c)$ and $||Au|| > \frac{b}{\gamma}$, then

$$\alpha(Au) = \min_{t \in [\xi_1, 1]} |Au(t)| \ge \gamma ||Au|| > b.$$

Therefore, condition (C_3) is also satisfied. By Theorem 1, there exist three positive solutions $v_1, v_2, v_3 \in K$ for the boundary value problem (6). Moreover, let

$$u_i(t) = A_{n-1} v_i(t) = \int_0^1 G_{n-1}(t,s) v_i(s) ds, \quad i = 1, 2, 3,$$

then u_1, u_2, u_3 are three positive solutions for the boundary value problem (1) such that

$$||u_1^{(2(n-1))}|| < a, \qquad b < \alpha(u_2^{(2(n-1))}),$$

and

$$||u_3^{(2(n-1))}|| > a, \quad \text{with} \quad \alpha(u_3^{(2(n-1))}) < b.$$

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