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A Coupled System With Integral Conditions*

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Abstract

We prove a result, which governs the separation of a system of two coupled equations, and this separation leads to study boundary value problems with non-local conditions. Using the Riesz representation theorem, we prove the existence and uniqueness of generalized solutions.

1 Introduction

Coupled Schrödinger equations [3, 4, 5, 8] are needed in the formulation of various physical situations but are usually not easy to handle and one must frequently have recourse to numerical treatments. Such system of equations becomes more complicated when studied with non-local boundary conditions. In the present work, a coupled system of two ordinary differential equations with integral conditions is considered. Here we have to mention that the non-local boundary value problems for second order differential equations are mainly motivated by the works of Bitsadze [1, 2] and were the subject of some recent papers (see, e.g. [6, 7]). We may say that the boundary value problems with non-local conditions for coupled system of differential equations constitute a very interesting and important class of problems. Motivated by this, we consider the following problem:

$$-\frac{d^2u}{dt^2}+p(t)\frac{du}{dt}+r(t)u = q_1(t)\left(v-\frac{dv}{dt}\right)+f_1(t) \tag{1}$$

$$-\frac{d^2v}{dt^2}+r(t)\frac{dv}{dt}+p(t)v = q_2(t)\left(u-\frac{du}{dt}\right)+f_2(t) \tag{2}$$

$$u(0) = 0, \quad \int_0^T u(t)dt=0 \tag{3}$$

$$v(0) = 0, \quad \int_0^T v(t)dt=0 \tag{4}$$

where $p(t)$, $q_1(t)$, $q_2(t)$ and $r(t)$ are assumed to be analytic functions, and $f_1,f_2\in L_2(0,T)$.

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We mention that the above system contains as a special case of the coupled Schrödinger equations [3, 4].

We first begin with the following result on the separation of this system.

LEMMA 1. The system (1)–(4) can always be decoupled without increase of the order of the differential equations if and only if the functions $q_1(t), q_2(t)$ are proportional to the difference $r(t) - p(t)$.

PROOF. Multiplying equation (1) by λ and adding equation (2) we get

$$\begin{aligned} & -\frac{d^2(u+\lambda v)}{dt^2} + [p+\lambda q_2]\frac{du}{dt} + [q_1+\lambda r]\frac{dv}{dt} + [r-\lambda q_2]\,u + [\lambda p - q_1]\,v \\ = \; & f_1 + \lambda f_2. \end{aligned} \tag{5}$$

It may be shown that the two equations (1)–(2) are separated if and only if the following condition is satisfied

$$q_2(t)\lambda^2 + [p(t) - r(t)]\,\lambda - q_1(t) = 0. \tag{6}$$

This condition means that λ is independent of t such that

$$\lambda_{1,2} = \frac{r-p}{2q_2} \pm \frac{1}{2}\left[\left(\frac{p-r}{q_2}\right)^2 + 4\frac{q_1}{q_2}\right]^{\frac{1}{2}}, \tag{7}$$

and (1)–(4) is separated in the forms

$$-\frac{d^2 w_1}{dt^2} + \Phi_1(t)\frac{dw_1}{dt} + \Psi_1(t)w_1 = F(t) \tag{8}$$

$$w_1(0) = 0, \; \int_0^T w_1(t)dt = 0, \tag{9}$$

and

$$-\frac{d^2 w_2}{dt^2} + \Phi_2(t)\frac{dw_2}{dt} + \Psi_2(t)w_2 = G(t) \tag{10}$$

$$w_2(0) = 0, \; \int_0^T w_2(t)dt = 0, \tag{11}$$

where

$$\begin{aligned} w_i(t) &= u(t) + \lambda_i v(t), \; i = 1, 2, \\ \Phi_1(t) &\equiv \Psi_2(t) = \frac{p(t)+r(t)}{2} + \frac{1}{2}\left[(p(t)-r(t))^2 + 4q_1(t)q_2(t)\right]^{\frac{1}{2}}, \\ \Psi_1(t) &\equiv \Phi_2(t) = \frac{p(t)+r(t)}{2} - \frac{1}{2}\left[(p(t)-r(t))^2 + 4q_1(t)q_2(t)\right]^{\frac{1}{2}}, \end{aligned}$$

and

$$F(t) = f_1(t) + \lambda_1 f_2(t), \quad G(t) = f_1(t) + \lambda_2 f_2(t).$$

We assume that the functions $\Phi_i(t)$ and $\Psi_1(t)$, $\Psi_2(t)$ are bounded on the interval $[0, T]$:

$$0 < \Psi_i(t) \le \alpha_i, 0 < \Phi_i(t) \le \sigma(t), \; i = 1, 2 \tag{12}$$

where $\sigma(t) = (T-t)/\sqrt{2}$.

2 Functional Spaces

The problems (8)–(9) and (10)–(11) will be considered as operator equations

$$\ell_1 w_1 = F, w_1 \in D(\ell_1),$$

and

$$\ell_2 w_2 = G, w_2 \in D(\ell_2)$$

respectively, where

$$\ell_i w_i \equiv -\frac{d^2 w_i}{dt^2} + \Phi_i(t)\frac{dw_i}{dt} + \Psi_i(t) w_i,\ i = 1, 2.$$

The domain $D(\ell_1)$ of the operator ℓ_1(respectively $D(\ell_2)$ of the operator ℓ_2) is given by $D(\ell_1) = H_0^2(0,T)$ (respectively $D(\ell_2) = H_0^2(0,T)$) the subspace of the Sobolev space $H^2(0,T)$, which consists of all the functions $w_1 \in H^2(0,T)$ (respectively $w_2 \in H^2(0,T)$) satisfying the conditions (9) (respectively (11)).

Let $H_\sigma^1(0,T)$ be the weighted Sobolev space defined as follows:

$$H_\sigma^1(0,T) = \left\{ w \mid w \in L_2(0,T), \sigma(t)\frac{dw}{dt} \in L_2(0,T), w(0) = \int_0^T w(t)dt = 0 \right\}$$

with

$$(w,z)_{1,\sigma} = \int_0^T w(t)z(t)dt + \int_0^T \sigma^2(t)\frac{dw}{dt}\frac{dz}{dt}dt,$$

and the finite norm

$$\| w \|_{1,\sigma}^2 = \int_0^T w^2(t)dt + \int_0^T \sigma^2(t)\left(\frac{dw}{dt}\right)^2 dt.$$

Define now the operator M by

$$Mz = (T-t)\int_0^t z(\tau)d\tau + \frac{1}{2}(T-t)^2 z(t),\ \forall z \in H_\sigma^1(0,T).$$

DEFINITION 1. A function $w_1 \in H_\sigma^1(0,T)$ (respectively $w_2 \in H_\sigma^1(0,T)$) is called a generalized solution of (8)–(9) (respectively (10)–(11)), if $(w_1, z_1)_{1,\sigma} = (\ell_1 w_1, M z_1)_{L_2}$ (respectively $(w_2, z_2)_{1,\sigma} = (\ell_2 w_2, M z_2)_{L_2}$) for all $z_1 \in H_\sigma^1(0,T)$ (respectively for all $z_2 \in H_\sigma^1(0,T)$).

3 Existence and Uniqueness Theorem

In order to prove the existence and uniqueness of a generalized solution of (1)–(4). We first study the following subsidiary problem of (8)–(9):

$$\ell_0 w_1 \equiv -\frac{d^2 w_1}{dt^2} = F(t), \tag{13}$$

$$w_1(0) = 0, \int_0^T w_1(t)dt = 0, \tag{14}$$

where ℓ_0 is the principal part of ℓ_1.

THEOREM 1. Let $F(t) \in L_2(0,T)$. Then there exists one and only one generalized solution $w_1 \in H^1_\sigma(0,T)$ of (13)–(14).

PROOF. Consider the scalar product $\left(-\frac{d^2w_1}{dt^2}, Mz_1\right)_{L_2}$, employing integration by parts and taking account of conditions (14), we obtain

$$\left(-\frac{d^2w_1}{dt^2}, Mz_1\right)_{L_2} = (w_1, z_1)_{1,\sigma}. \tag{15}$$

Let $F(t) \in L_2(0,T)$, then $\zeta(z_1) = (F, Mz_1)_{L_2}$ is a bounded linear functional on $H^1_\sigma(0,T)$. Indeed,

$$|\zeta(z_1)| = |(F, Mz_1)_{L_2}| \leq ||F||_{L_2}||Mz_1||_{L_2}. \tag{16}$$

For the function $z_1 \in H^1_\sigma(0,T)$, we have the following Poincaré's estimates

$$\int_0^T z_1^2(t)dt \leq 4\int_0^T (T-t)^2 \left(\frac{dz_1(t)}{dt}\right)^2 dt, \tag{17}$$

and

$$\int_0^T \left[\int_0^t z_1(\tau)d\tau\right]^2 dt \leq 4T^2 \int_0^T z_1^2(t)dt. \tag{18}$$

Using inequalities (17)–(18), we obtain

$$|\zeta(z_1)| \leq 2\sqrt{2}T^2||F||_{L_2}||z_1||_{1,\sigma}.$$

Thus, by Riesz's Representation Theorem there exists one and only one generalized solution $w_1 \in H^1_\sigma(0,T)$ such that

$$\zeta(z_1) = (F, Mz_1)_{L_2} = (w_1, z_1)_{1,\sigma}, \ \forall z_1 \in H^1_\sigma(0,T).$$

i.e., $w_1 \in H^1_\sigma(0,T)$ is a generalized solution of (13)–(14).

COROLLARY 1. We have the following *a priori* estimate

$$c_1||w_1||_{1,\sigma} \leq ||\ell_0 w_1||_{L_2}, \tag{19}$$

where $c_1 > 0$ is independent on w_1.

Indeed, letting $\frac{1}{c_1} = 2\sqrt{2}T^2$, we obtain the inequality (19).

Now, consider the general case. The idea in the proof is to derive the results for the equation $\ell_1 w_1 = F$ with integral conditions (9) from the results for $\ell_0 w_1 = F$ by means of continuous variation of the parameter μ [9]. Consider the operator ℓ_μ, which has the same domain of definition as ℓ_1, and coincides with ℓ_1 for $\mu = 1$, however, ℓ_0 contains only the principal part of ℓ_1. From the definition of ℓ_μ we get the formula:

$$\ell_\mu = \ell_{\mu_0} + (\mu - \mu_0)(\ell_1 - \ell_0), \mu, \ \mu_0 \in (0,1). \tag{20}$$

We need the following Lemmas.

LEMMA 2. The operator $\ell_1 - \ell_0 : D(\ell_1) \subset H_\sigma^1 \to L_2$ is bounded, that is, there exists a positive constant c_2, which does not depend on w_1, such that

$$\| (\ell_1 - \ell_0)w_1 \|_{L_2} \leq c_2 \| w_1 \|_{1,\sigma}, \ \forall w_1 \in H_0^2(0,T).$$

Indeed, this inequality is direct consequence of conditions (12).

LEMMA 3. There exists a constant c_3 which does not depend on μ nor w_1, such that

$$\| w_1 \|_{1,\sigma} \leq c_3 \| \ell_\mu w_1 \|_{L_2}, \ \forall w_1 \in H_0^2(0,T).$$

PROOF. From the definition of ℓ_μ and Corollary 1, we have

$$c_1 ||w_1||_{1,\sigma} \leq ||\ell_\mu w_1||_{L_2} + \mu \| (\ell_1 - \ell_0)w_1 \|_{L_2} .$$

Using Lemma 2 we obtain

$$c_1 ||w_1||_{1,\sigma} \leq ||\ell_\mu w_1||_{L_2} + \mu c_2 \| w_1 \|_{1,\sigma} .$$

Choosing $c(\mu) = \frac{1}{c_1 - \mu c_2}$, where $c_1 - \mu c_2 > 0$, we see that for $\mu \in (0,1)$, there exists a constant $c(\mu)$ such that

$$||w_1||_{1,\sigma} \leq c(\mu) ||\ell_\mu w_1||_{L_2} \text{ for all } w_1 \in H_0^2(0,T).$$

On the basis on the last inequality, Lemma 3 can be proved in a similar way with the techniques considered in the proof of Lemma II.3 in [9].

It can be proved in the standard way that the operator ℓ_1 from H_σ^1 to L_2 is pre-closed. It is well known that ℓ_1 is pre-closed if and only if

$$(w_1^n \to 0 \text{ in } H_\sigma^1 \text{ and } \ell_1 w_1^n \to F \ \text{ in } L_2 \text{ as } n \to \infty) \Longrightarrow F = 0.$$

LEMMA 4. We have $\Re(\overline{\ell_\mu}) = \overline{\Re(\ell_\mu)}$, where $\overline{\ell_\mu}$ and $\overline{\Re(\ell_\mu)}$ stand for the closure of ℓ_μ and $\Re(\ell_\mu)$ respectively.

PROOF. It follows from the definition of $\overline{\ell_\mu}$ that $\Re(\overline{\ell_\mu}) \subset \overline{\Re(\ell_\mu)}$. It remains to prove the opposite inclusion. Suppose that $F \in \overline{\Re(\ell_\mu)}$, then there exists a sequence $(w_{1,n}) \subseteq D(\ell_\mu)$ such that $\ell_\mu w_{1,n} \to F$ as $n \to \infty$. According to Lemma 3 we have

$$||w_{1,m} - w_{1,n}||_{1,\sigma} \leq c_3 ||\ell_\mu w_{1,m} - \ell_\mu w_{1,n}||_{L_2} \to 0$$

as $m, n \to \infty$. We conclude that $(w_{1,n})$ is a Cauchy sequence in the space $H_\sigma^1(0,T)$ and converges to an element $w_1 \in H_\sigma^1(0,T)$, and $\overline{\ell_\mu} w_1 = F$.

LEMMA 5. Assume that $\Re(\overline{\ell_{\mu_0}}) = L_2(0,T)$. Then

$$\| \left(\overline{\ell_{\mu_0}}\right)^{-1} \|_{1,\sigma} \leq c_3$$

and

$$\| \left(\overline{\ell_{\mu_0}}\right)^{-1} (\overline{\ell_1 - \ell_0}) w_1 \|_{1,\sigma} \leq c_4 \| w_1 \|_{1,\sigma},$$

where $c_4 = c_2 \times c_3$.

Indeed, these statements follow from Lemma 3 and Lemma 2 respectively.

To conclude our paper, we give the following important result.

THEOREM 2. For every $F \in L_2(0,T)$, the equation $\overline{\ell_1} w_1 = F$ has a unique generalized solution $w_1 \in H^1_\sigma(0,T)$.

PROOF. We consider the equation

$$\overline{\ell_\mu} w_1 = F \tag{21}$$

From (20) equation (21) can be written as:

$$\overline{\ell_\mu} w_1 = \overline{\ell_{\mu_0}} w_1 + (\mu - \mu_0)(\overline{\ell_1 - \ell_0}) w_1 = F, \ \ \mu, \mu_0 \in (0,1). \tag{22}$$

Now assume that we already know that $\Re(\overline{\ell_{\mu_0}}) = L_2(0,T)$. A solution of the equation

$$w_1 + (\mu - \mu_0) \left(\overline{\ell_{\mu_0}}\right)^{-1} (\overline{\ell_1 - \ell_0}) w_1 = \left(\overline{\ell_{\mu_0}}\right)^{-1} F, \mu, \mu_0 \in (0,1), \tag{23}$$

is then also a solution of (22) and therefore is a solution of (21) as well. Let $\mid \mu - \mu_0 \mid < \frac{1}{c_4}$, with $B = (\mu - \mu_0) \left(\overline{\ell_{\mu_0}}\right)^{-1} (\overline{\ell_1 - \ell_0})$ and $\xi = \left(\overline{\ell_{\mu_0}}\right)^{-1} F$, then (23) can be written as

$$w_1 + B w_1 = \xi.$$

Here

$$||B||_{1,\sigma} = \sup_{w_1 \in H^1_\sigma} \frac{||B w_1||_{1,\sigma}}{||w_1||_{1,\sigma}} \leq c_4 \mid \mu - \mu_0 \mid < 1.$$

The Neumann series

$$w_1 = \sum_{k=0}^{\infty} (-B)^k \xi$$

is then a solution to equation (23).

We have thus proved that if $\Re(\overline{\ell_{\mu_0}}) = L_2(0,T)$ and $\mid \mu - \mu_0 \mid < \frac{1}{c_4}$, then $\Re(\overline{\ell_\mu}) = L_2(0,T)$. Proceeding step by step in this way, this gives in a finite number of steps that $\Re(\overline{\ell_\mu}) = L_2(0,T)$.

REMARK. A similar procedure can be applied to prove Theorems 1–2 for the problem (10)–(11).

COROLLARY 2. Under the hypothesis of Lemma 1, the problem (1)–(4) has one and only one generalized solution

$$\{u, v\} \in H^1_\sigma(0,T) \times H^1_\sigma(0,T).$$

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