Applied Mathematics E-Notes, 4(2004), 74-79 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Qualitative Analysis For A Class Of Plane Systems*†

Zhi-cong Liu and Bei-ye Feng ‡

Received 20 September 2003

Abstract

In this paper, we give a rigorous mathematical and parameter analysis for a plane system and obtain the conditions and parameters region for global existence and uniqueness as well as the the global bifurcation diagram of limit cycles. Our results are applied to a rheodynamic model.

1 Introduction

In this paper, we consider the system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -x + f(y). \end{cases} \tag{1}$$

There are several studies related to the above system, see for examples [1-6]. Here, we give a rigorous qualitative analysis and complete parameter analysis to prove the global existence, uniqueness and stability of limit cycles, and present the parameter region where there are no periodic solutions. In other words, the global bifurcation diagram of limit cycles will be obtained. We also apply our results to a rheodynamic model of cardiac.

2 Stability of equilibrium point

Let $f(y)$ satisfy the following conditions:

(1) $f(0) = 0, f(+\infty) = -\infty, f(-\infty) = +\infty$;
(2) $f'(y) = 0$ has at most two real roots;
(3) $\frac{f(y)}{y} \to -\infty$ as $y \to +\infty$;
(4) $f'''(0) < 0$; and
(5) if $f'(0) > 0$ and $y \neq 0$, then $f'(y) - yf''(y) \geq 0$.

It is easy to see that $(0,0)$ is the only equilibrium point of (1). The Jacobian matrix at $(0,0)$ of (1) is

$$D_xF(0) = A = \begin{pmatrix} 0 & 1 \\ -1 & f'(0) \end{pmatrix},$$

*Mathematics Subject Classifications: 34C05, 34C23, 34C37.

†Supportedby the National Natural Science Foundation of China (No.10171099)

‡Institute of Applied Mathematics, Academy of Mathematics and Systems Sciences, Beijing 100080, P. R. China

and

$$\det A = 1 > 0.$$

Let $D = \det A$ and $T = \text{Trace}A$. Then the characteristic equation is

$$\lambda^2 - T\lambda + D = 0.$$

Accordingly,

(I) if $T > 0$, then $(0,0)$ is not stable;

(II) if $T < 0$, then $(0,0)$ is stable; and

(III) if $T = 0$, then $(0,0)$ may be a center or a focus.

We apply the method in [7] and [8]: Let

$$V = x^2 + y^2 - \frac{2}{3}f''(0)x^3 - f''(0)xy^2 - \frac{1}{4}f''(0)^2y^4 - \frac{1}{8}f'''(0)x^3y - \frac{5}{24}f'''(0)xy^3$$

$V(x,y)$ is positive definite in some sufficiently small neighborhood of $(0,0)$. Compute the derivative of V through the orbit of (1):

$$\dot{V}|_{(1)} = \frac{1}{8}f'''(0)(x^2+y^2)^2 + O(5)$$

Since $f'''(0) < 0$, so if $|x|$ and $|y|$ are sufficiently small, then $\dot{V}|_{(1)} < 0$. Therefore $(0,0)$ is an asymptotically stable focus.

By the above discussion, we have the following.

LEMMA 1. If $f^{'}(0) \leq 0$, then $(0,0)$ is asymptotically stable , else $(0,0)$ is not stable.

3 Positive invariant set

Consider the region **D** bounded by the line segments

$$\begin{aligned}
&l^+ : y = m, -m \leq x \leq 0,\\
&l^- : y = -m, 0 \leq x \leq m,\\
&l_r : x = m, -m \leq y \leq 0,\\
&l_l : x = -m, 0 \leq y \leq m,\\
&l_1 : y = -x + m, 0 \leq x \leq m,\\
&l_2 : y = -x - m, -m \leq x \leq 0,
\end{aligned}$$

where $m > 0$. Denote

$$\begin{aligned}
y^+ &= \max\{y|f(y) = -y\},\\
y^- &= \min\{y|f(y) = -y\},\\
x^+ &= \max\{f(y) + 2y|y \geq 0\},\\
x^- &= \min\{f(y) + 2y|y \leq 0\}.
\end{aligned}$$

It is easy to verify that the definitions are well-defined.

LEMMA 2. If $m > \max\{y^+, -y^-, x^+, -x^-, 0\}$, then **D** is the positive invariant set of system (1).

PROOF. It is sufficient to show that the vector field on the boundary of $\mathbf{D}$ defined by system (1) points towards the interior of $\mathbf{D}$. (i) On l^+, because $m > y^+ = \max\{y|-y = f(y)\}$, so $x > f(m)$ when $-m \leq x \leq 0$. So $\dot{y}|_{l^+} = -x + f(y) < 0$ for $-m \leq x \leq 0$. In a similar manner, on l^-, because $-m < y^- = \min\{y| - y = f(y)\}$, so $x < f(-m)$ for $0 \leq x \leq m$. So $\dot{y}|_l^- = -x + f(y) > 0$ for $0 \leq x \leq m$. (ii) On l_l, $\dot{x}|_{l_l} = y > 0$, for $0 \leq y \leq m$. On l_r, $\dot{x}|_{l_r} = y < 0$ for $-m \leq y \leq 0$. (iii) For l_1, its external normal vector is $V_{l_1} = (1, 1)$, so the inner product of V_{l_1} and the vector field defined by system (1) is

$$P = (1,1) \bullet (\dot{x}, \dot{y})|_{l_1} = \dot{x} + \dot{y} = -x + f(y) + y|_{l_1} = f(y) + 2y - m$$

for $0 \leq y \leq m$. By $m > x^+ = \max\{f(y) + 2y|y \geq 0\}$, we have $P = f(y) + 2y - m < 0$ on l_1. On l_2, in a similar manner,

$$P = (-1,-1) \bullet (\dot{x}, \dot{y})|_{l_2} = -(\dot{x} + \dot{y}) = x - f(y) - y|_{l_1} = -f(y) - 2y - m,$$

because $-m > x^- = \min\{f(y) + 2y|y \leq 0\}$, so $P = -m - [f(y) + 2y] < 0$. This means that the vector field also points towards the interior of $\mathbf{D}$ on l_2. So $\mathbf{D}$ is the positive invariant set of system (1). The proof is complete.

By Lemma1 and Lemma2, we have the following.

LEMMA 3. If $f'(0) > 0$, then the system (1) has at least one stable inner limit cycle and one stable outer limit cycle (they may be coincident).

4 The uniqueness and bifurcation of limit cycles

We have the following.

THEOREM 1. If $f'(0) > 0$, then the system (1) has a unique limit cycle which is also stable.

PROOF. Let $x = -\bar{y}$ and $y = \bar{x}$. Then (1) is transformed into

$$\begin{cases} \frac{d\bar{x}}{dt} = \bar{y} + f(\bar{x}), \\ \frac{d\bar{y}}{dt} = -\bar{x}. \end{cases} \tag{2}$$

We have $-\frac{\partial f(\bar{x})}{\partial \bar{x}} \in C^0(-\infty, +\infty)$ and $-\frac{\partial f(\bar{x})}{\partial \bar{x}}|_{\bar{x}=0} = -\lambda < 0$. So when $\bar{x} \neq 0$, we have

$$\frac{\partial}{\partial \bar{x}}\left(\frac{-\frac{f(\bar{x})}{\bar{x}}}{\bar{x}}\right) = \frac{f^{'}(\bar{x}) - \bar{x}f^{''}(\bar{x})}{\bar{x}^2} \geq 0$$

By [3], the limit cycle in the system (2) (if it exists) is unique.

LEMMA 4. If $f'(0) \leq 0$, then the system (1) has no limit cycles.

PROOF. Under the conditions satisfied by $f(y)$, $f(y)$ can take the following form

$$f(y) = p(y)(-y^3 + ay^2 + \lambda y) = G(y, \lambda)$$

where $p(y) > 0$ for any y. It is easy to see that $f'(0) = p(0)\lambda$. Take λ as the parameter for the systems. For any $\lambda \in (+\infty, -\infty)$, the equilibrium of (1) is preserved, and for any fixed point (x, y), and any $\lambda_1, \lambda_2 \in (-\infty, +\infty)$ satisfying $\lambda_1 > \lambda_2$, we have

$$\begin{vmatrix} P(x,y,\lambda_2) & Q(x,y,\lambda_2) \\ P(x,y,\lambda_1) & Q(x,y,\lambda_1) \end{vmatrix} = \begin{vmatrix} y & -x + G(y,\lambda_1) \\ y & -x + G(y,\lambda_2) \end{vmatrix} = p(y)(\lambda_1 - \lambda_2)y^2 \geq 0$$

where the equality cannot be satisfied on a global closed orbits of (1). So the set of points $(P(x, y, \lambda), Q(x, y, \lambda))$ over $\lambda \in (-\infty, +\infty)$ generates a general rotational vector field.

Assume that when $\lambda = 0$, the system (1) has limit cycles. Because the equilibrium is asymptotically stable, and the system has a positive invariant set $\mathbf{D}$, so the system either has at least one inner unstable limit cycle or has a stable limit cycle and an unstable limit cycle. Suppose the former case holds. As λ increases, the limit cycle breaks down into more than two cycles (see [1]), but by Theorem 1, if $\lambda > 0$, then (1) have a unique stable limit cycle Γ, which is a contradiction. Suppose the latter case holds. As λ increases, there exists $\epsilon > 0$ such that for $\lambda \in (0, \epsilon)$, the limit cycles do not vanish, but extends or contracts monotonously depending on λ, which is contrary to the uniqueness of the limit cycle for the system with $\lambda > 0$. So when $\lambda = 0$, the system does not have any limit cycle.

Assume that when $\lambda < 0$, there exists a limit cycle Γ_-. Since the equilibrium of the system is globally attractive as $\lambda = 0$, so the orbits of the system with $\lambda = 0$ must spiral clockwise and pass through Γ_- from outside to inside. Clearly, this is contrary to the condition

$$\begin{vmatrix} P(x, y, \lambda_2) & Q(x, y, \lambda_2) \\ P(x, y, \lambda_1) & Q(x, y, \lambda_1) \end{vmatrix} \geq 0.$$

So when $\lambda < 0$, the system does not have any limit cycle. In other words, if $f^{'}(0) \leq 0$, the system (1) does not have any limit cycles.

From Lemma 3 and Theorem 1, we know that when $f'(0) > 0$, the limit cycle in the system (1) must be asymptotically stable.

We summarize our discussions as follows.

THEOREM 2. If $f^{'}(0) > 0$, then the system (1) has a unique limit cycle, and the limit cycle is asymptotically stable; if $\lambda = f^{'}(0) \leq 0$, then the system (**??**) does not have any limit cycles.

5 Application

In [10], Valko and Nikolov propose a mathematical model about cardiac pulsations which can explain many physiology phenomena such as sudden death of heart. But in [2], they have only made Hopf analysis, so the result that can be obtained is only the existence of periodic solution on a small scale. The global existence, uniqueness and nonexistence of limit cycles were not discussed. In this section, we will apply the previous results to the model, and give a rigorous qualitative analysis and complete parameter analysis for the system, proving the global existence, uniqueness and stability of limit cycle , and present the parameter region where there are no periodic solutions. The model proposed in [7] can be taken in the form

$$\begin{cases} \dot{x} = y \\ \dot{y} = -gx + (a - 3bc^2)y + 3bcy^2 - by^3 \end{cases} \tag{3}$$

where $a, b > 0$, $c > 0$, $g > 0$ are parameters. Let

$$\begin{array}{l} x = \frac{\bar{x}}{\sqrt{g}} \\ y = \bar{y} \end{array}$$

Then (3) is changed into

$$\begin{cases} \frac{d\bar{x}}{dt} = \sqrt{g}\bar{y} \\ \frac{d\bar{y}}{dt} = -\sqrt{g}\bar{x} + (a - 3bc^2)\bar{y} + 3bc\bar{y}^2 - b\bar{y}^3 \end{cases} . \tag{4}$$

Let $t = \frac{\tau}{\sqrt{g}}$, then (4) is changed into

$$\begin{cases} \frac{d\bar{x}}{d\tau} = \bar{y} \\ \frac{d\bar{y}}{dt} = -\bar{x} + \frac{(a-3bc^2)}{\sqrt{g}}\bar{y} + \frac{3bc}{\sqrt{g}}\bar{y}^2 - \frac{b}{\sqrt{g}}\bar{y}^3 \end{cases} . \tag{5}$$

So

$$f(\bar{y}) = \frac{(a - 3bc^2)}{\sqrt{g}}\bar{y} + \frac{3bc}{\sqrt{g}}\bar{y}^2 - \frac{b}{\sqrt{g}}\bar{y}^3,$$

$$f'(\bar{y}) = \frac{(a - 3bc^2)}{\sqrt{g}} + \frac{6bc}{\sqrt{g}}\bar{y} - \frac{3b}{\sqrt{g}}\bar{y}^2,$$

$$f''(\bar{y}) = \frac{6bc}{\sqrt{g}} - \frac{6b}{\sqrt{g}}\bar{y}$$

and

$$f'''(\bar{y}) = -\frac{6b}{\sqrt{g}}.$$

It is easy to verify that $f(0) = 0$, $f(+\infty) = -\infty$, $f(-\infty) = +\infty$; that the equation $f'(\bar{y}) = 0$ has at most two real roots; that

$$\lim_{\bar{y}\to+\infty} \frac{f(\bar{y})}{\bar{y}} \to -\infty;$$

$$f'''(0) = -\frac{6b}{\sqrt{g}} < 0;$$

and

$$f'(y) - yf''(y) = \frac{3b}{\sqrt{g}}y^2 + \frac{a - 3bc^2}{\sqrt{g}}.$$

Thus if $f'(0) = \frac{a-3bc^2}{\sqrt{g}} > 0$, then $f'(y) - yf''(y) \geq 0$.

In view of Theorem 2, we have the following corollary.

THEOREM 3. If $a - 3bc^2 > 0$, then the system (3) has a unique limit cycle, and the limit cycle is asymptotically stable; if $a - 3bc^2 \leq 0$, then the system (3) has no limit cycles.

References

[1] S. H. Ding, Theorem for the existence of n limit cycles for the Liénard equation on finite interval, Scientia Sinica, 25(9)(1982), 792–800.

[2] G. F. D. Duff and N. Levinson, On the non-uniqueness of periodic solutions for an asymmetric Liénard equation, Quart. Appl. Math, 10(1952), 86–88.

[3] Y. Q. Ye and L. S. Chen, On the uniqueness of limit cycle of the system of differential equations, Acta Math. Sinica, 18(1975), 219–222.

[4] Y. Q. Ye, Theory of Limit Cycle, 2nd edition, Shanghai Scince and Technology Press, Shanghai, 1984.

[5] Z. F. Zhang and Q. M. He, A sufficient condition for the existence of no more than n limit cycles for the Liénard equation, Acta Math. Sinica, 25(1982), 585–594.

[6] Z. F. Zhang, Theorem about the existence of n limit cycles, Scientia Sinica, 10(1980), 941-948.

[7] J. Y. Zhang and B. Y. Feng, The Geometrical Theory and Bifurcation Problem of Ordinary Differential Equation, 2nd edition, Peking University Press, Beijing, 2000.

[8] J. B. Li and B. Y. Feng, Stability, Bifurcations and Chaos, Yunnan Science and Technology Press, Kunming, 1995.

[9] Z. F. Zhang, T. R. Ding, W. Z. Huang and Z. X. Dong, Qualitative Theory of Differential Equations, Translations of Mathematical Monographs, 101, American Mathematical Society, Providence, RI, 1992.

[10] V. G. Petrov, S. G. Nikolov, Rheodynamic model of cardiac pressure pulsations, Math. Biosciences 157(1999), 237–252.