

# Gauge Symmetries And Noether Currents In Optimal Control \*

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## Abstract

We extend the second Noether theorem to optimal control problems which are invariant under symmetries depending upon  $k$  arbitrary functions of the independent variable and their derivatives up to some order  $m$ . As far as we consider a semi-invariance notion, and the transformation group may also depend on the control variables, the result is new even in the classical context of the calculus of variations.

## 1 Introduction

The study of invariant variational problems

$$\text{Minimize } J[x(\cdot)] = \int_a^b L(t, x(t), \dot{x}(t)) dt$$

in the calculus of variations was initiated in the early part of the XX century by Emmy Noether who, influenced by the works of Klein and Lie on the transformation properties of differential equations under continuous groups of transformations (see e.g. [2, Ch. 2]), published in her seminal paper [13, 14] of 1918 two fundamental theorems, now classical results and known as the (first) Noether theorem and the second Noether theorem, showing that invariance with respect to a group of transformations of the variables  $t$  and  $x$  implies the existence of certain conserved quantities. These results, also known as *Noether's symmetry theorems*, have profound implications in all physical theories, explaining the correspondence between symmetries of the systems (between the group of transformations acting on the independent and dependent variables of the system) and the existence of conservation laws. This remarkable interaction between the concept of invariance in the calculus of variations and the existence of first integrals (Noether currents) was clearly recognized by Hilbert [6] (cf. [12]).

The first Noether theorem establishes the existence of  $\rho$  first integrals of the Euler-Lagrange differential equations when the Lagrangian  $L$  is invariant under a group of transformations containing  $\rho$  parameters. This means that the invariance hypothesis

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leads to quantities which are constant along the Euler-Lagrange extremals. Extensions for the Pontryagin extremals of optimal control problems are available in [19, 21, 20].

The second Noether theorem establishes the existence of  $k(m+1)$  first integrals when the Lagrangian is invariant under an infinite continuous group of transformations which, rather than dependence on parameters, as in the first theorem, depend upon  $k$  arbitrary functions and their derivatives up to order  $m$ . This second theorem is not as well known as the first. It has, however, some rather interesting implications. If for example one considers the functional of the basic problem of the calculus of variations in the autonomous case,

$$J[x(\cdot)] = \int_a^b L(x(t), \dot{x}(t)) dt, \quad (1)$$

the classical Weierstrass necessary optimality condition can easily be deduced from the fact that the integral (1) is invariant under transformations of the form  $T = t + p(t)$ ,  $X = x(t)$ , for an arbitrary function  $p(\cdot)$  (see [11, p. 161]). The second Noether theorem is related to: (i) parameter invariant variational problems, i.e., problems of the calculus of variations, as in the homogeneous-parametric form, which are invariant under arbitrary transformations of the independent variable  $t$  (see [1, p. 266], [11, Ch. 8], [3, p. 179]); (ii) the singular Lagrangians and the constraints in the Hamiltonian formalism, a framework studied by Dirac-Bergmann (see [4, 5]); (iii) the physics of gauge theories, such as the gauge transformations of electrodynamics, electromagnetic field, hydromechanics, and relativity (see [3, pp. 186–189], [11, p. 160], [10], [17]). For example, if the Lagrangian  $L$  represents a charged particle interacting with a electromagnetic field, one finds that it is invariant under the combined action of the so called gauge transformation of the first kind on the charged particle field, and a gauge transformation of the second kind on the electromagnetic field. As a result of this invariance it follows, from second Noether's theorem, the very important conservation of charge. The invariance under gauge transformations is a basic requirement in Yang-Mills field theory, an important subject, with many questions for mathematical understanding (cf. [7]).

To our knowledge, no second Noether type theorem is available for the optimal control setting. One such generalization is our concern here. Instead of using the original argument [13, 14] of Emmy Noether, which is fairly complicated and depends on some deep and conceptually difficult results in the calculus of variations, our approach follows, *mutatis mutandis*, the paper [19], where the first Noether theorem is derived almost effortlessly by means of elementary techniques, with a simple and direct approach, and it is motivated by the novelties introduced by the author in [21]. Even in the classical context (cf. e.g. [10]) and in the simplest possible situation, for the basic problem of the calculus of variations, our result is new since we consider symmetries of the system which alter the cost functional up to an exact differential; we introduce a semi-invariant notion with some weights  $\lambda^0, \dots, \lambda^m$  (possible different from zero); and our transformation group may depend also on  $\dot{x}$  (the control). Our result hold both in the normal and abnormal cases.

## 2 The Optimal Control Problem

We consider the optimal control problem in Lagrange form on the compact interval  $[a, b]$ :

$$\text{Minimize } J[x(\cdot), u(\cdot)] = \int_a^b L(t, x(t), u(t)) dt$$

over all admissible pairs  $(x(\cdot), u(\cdot))$ <sup>1</sup>

$$(x(\cdot), u(\cdot)) \in W_{1,1}^n([a, b]; R^n) \times L_\infty^r([a, b]; \Omega \subseteq R^r),$$

satisfying the control equation

$$\dot{x}(t) = \varphi(t, x(t), u(t)) \quad \text{a.e. } t \in [a, b].$$

The functions  $L : R \times R^n \times R^r \rightarrow R$  and  $\varphi : R \times R^n \times R^r \rightarrow R^n$  are assumed to be  $C^1$  with respect to all variables and the set  $\Omega$  of admissible values of the control parameters is an arbitrary open set of  $R^r$ .

Associated to the optimal control problem there is the Pontryagin Hamiltonian  $H : [a, b] \times R^n \times \Omega \times R \times (R^n)^T \rightarrow R$  which is defined as

$$H(t, x, u, \psi_0, \psi) = \psi_0 L(t, x, u) + \psi \cdot \varphi(t, x, u). \quad (2)$$

A quadruple  $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$ , with admissible  $(x(\cdot), u(\cdot))$ ,  $\psi_0 \in R_0^-$ , and  $\psi(\cdot) \in W_{1,1}([a, b]; R^n)$  ( $\psi(t)$  is a covector  $1 \times n$ ), is called a *Pontryagin extremal* if the following two conditions are satisfied for almost all  $t \in [a, b]$ :

The Adjoint System

$$\dot{\psi}(t) = -\frac{\partial H}{\partial x}(t, x(t), u(t), \psi_0, \psi(t)); \quad (3)$$

The Maximality Condition

$$H(t, x(t), u(t), \psi_0, \psi(t)) = \max_{u \in \Omega} H(t, x(t), u, \psi_0, \psi(t)). \quad (4)$$

The Pontryagin extremal is called normal if  $\psi_0 \neq 0$  and abnormal otherwise. The celebrated *Pontryagin Maximum Principle* asserts that if  $(x(\cdot), u(\cdot))$  is a minimizer of the problem, then there exists a nonzero pair  $(\psi_0, \psi(\cdot))$  such that  $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$  is a Pontryagin extremal. Furthermore, the Pontryagin Hamiltonian along the extremal is an absolutely continuous function of  $t$ ,

$$t \mapsto H(t, x(t), u(t), \psi_0, \psi(t)) \in W_{1,1}([a, b]; R),$$

and satisfies the equality

$$\frac{dH}{dt}(t, x(t), u(t), \psi_0, \psi(t)) = \frac{\partial H}{\partial t}(t, x(t), u(t), \psi_0, \psi(t)), \quad (5)$$

for almost all  $t \in [a, b]$ , where on the left-hand side we have the total derivative with respect to  $t$  and on the right-hand side the partial derivative of the Pontryagin Hamiltonian with respect to  $t$  (cf. [16]. See [18] for some generalizations of this fact).

<sup>1</sup>The notation  $W_{1,1}$  is used for the class of absolutely continuous functions, while  $L_\infty$  represents the class of measurable and essentially bounded functions.

### 3 Main Result

To formulate a second Noether theorem in the optimal control setting, first we need to have appropriate notions of *invariance* and *Noether current*. We propose the following ones.

DEFINITION 1. A function  $C(t, x, u, \psi_0, \psi)$  which is constant along every Pontryagin extremal  $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$  of the problem,

$$C(t, x(t), u(t), \psi_0, \psi(t)) = k, \quad t \in [a, b], \quad (6)$$

for some constant  $k$ , will be called a *Noether current*. The equation (6) is the *conservation law* corresponding to the Noether current  $C$ .

DEFINITION 2. Let  $C^m \ni p : [a, b] \rightarrow R^k$  be an arbitrary function of the independent variable. Using the notation

$$\alpha(t) \doteq (t, x(t), u(t), p(t), \dot{p}(t), \dots, p^{(m)}(t)),$$

we say that the optimal control problem is *semi-invariant* if there exists a  $C^1$  transformation group

$$\begin{aligned} g & : [a, b] \times R^n \times \Omega \times R^{k*(m+1)} \rightarrow R \times R^n \times R^r, \\ g(\alpha(t)) & = (T(\alpha(t)), X(\alpha(t)), U(\alpha(t))), \end{aligned} \quad (7)$$

which for  $p(t) = \dot{p}(t) = \dots = p^{(m)}(t) = 0$  corresponds to the identity transformation,  $g(t, x, u, 0, 0, \dots, 0) = (t, x, u)$  for all  $(t, x, u) \in [a, b] \times R^n \times \Omega$ , satisfying the equations

$$\begin{aligned} & L(g(\alpha(t))) \frac{d}{dt} T(\alpha(t)) \\ & = \left( \lambda^0 \cdot p(t) + \lambda^1 \cdot \dot{p}(t) + \dots + \lambda^m \cdot p^{(m)}(t) \right) \frac{d}{dt} L(t, x(t), u(t)) \\ & + L(t, x(t), u(t)) + \frac{d}{dt} F(\alpha(t)) \end{aligned} \quad (8)$$

$$\frac{d}{dt} X(\alpha(t)) = \varphi(g(\alpha(t))) \frac{d}{dt} T(\alpha(t)), \quad (9)$$

for some function  $F$  of class  $C^1$  and for some  $\lambda^0, \dots, \lambda^m \in R^k$ . In this case the group of transformations  $g$  will be called a *gauge symmetry* of the optimal control problem.

REMARK 1. We use the term “gauge symmetry” to emphasize the fact that the group of transformations  $g$  depend on arbitrary functions. The terminology takes origin from gauge invariance in electromagnetic theory and in Yang-Mills theories, but it refers here to a wider class of symmetries.

REMARK 2. The identity transformation is a gauge symmetry for any given optimal control problem.

THEOREM 1 (second Noether theorem for Optimal Control). If the optimal control problem is semi-invariant under a gauge symmetry (7), then there exist  $k(m+1)$

Noether currents of the form

$$\begin{aligned} & \psi_0 \left( \frac{\partial F(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \lambda_j^i L(t, x(t), u(t)) \right) + \psi(t) \cdot \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \\ & - H(t, x(t), u(t), \psi_0, \psi(t)) \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \end{aligned}$$

( $i = 0, \dots, m, j = 1, \dots, k$ ), where  $H$  is the corresponding Pontryagin Hamiltonian (2).

REMARK 3. We are using the standard convention that  $p^{(0)}(t) = p(t)$ , and the following notation for the evaluation of a term:

$$(*)|_0 \doteq (*)|_{p(t)=\dot{p}(t)=\dots=p^{(m)}(t)=0}.$$

REMARK 4. For the basic problem of the calculus of variations, i.e., when  $\varphi = u$ , Theorem 1 coincides with the classical formulation of the second Noether theorem if one puts  $\lambda^i = 0$ ,  $i = 0, \dots, m$ , and  $F \equiv 0$  in the Definition 2, and the transformation group  $g$  is not allowed to depend on the derivatives of the state variables (on the control variables). In §4 we provide an example of the calculus of variations for which our result is applicable while previous results are not.

PROOF. Let  $i \in \{0, \dots, m\}$ ,  $j \in \{1, \dots, k\}$ , and  $(x(\cdot), u(\cdot), \psi_0, \psi(\cdot))$  be an arbitrary Pontryagin extremal of the optimal control problem. Since it is assumed that to the values  $p(t) = \dot{p}(t) = \dots = p^{(m)}(t) = 0$  it corresponds the identity gauge transformation, differentiating (8) and (9) with respect to  $p_j^{(i)}$  and then setting  $p(t) = \dot{p}(t) = \dots = p^{(m)}(t) = 0$  one gets:

$$\begin{aligned} & \lambda_j^i \frac{d}{dt} L + \frac{d}{dt} \frac{\partial F(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 = \frac{\partial L}{\partial t} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \frac{\partial L}{\partial x} \cdot \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \\ & + \frac{\partial L}{\partial u} \cdot \frac{\partial U(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + L \frac{d}{dt} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0, \end{aligned} \quad (10)$$

$$\begin{aligned} & \frac{d}{dt} \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 = \frac{\partial \varphi}{\partial t} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \frac{\partial \varphi}{\partial x} \cdot \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \\ & + \frac{\partial \varphi}{\partial u} \cdot \frac{\partial U(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \varphi \frac{d}{dt} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0, \end{aligned} \quad (11)$$

with  $L$  and  $\varphi$ , and its partial derivatives, evaluated at  $(t, x(t), u(t))$ . Multiplying (10) by  $\psi_0$  and (11) by  $\psi(t)$ , we can write:

$$\psi_0 \left( \frac{\partial L}{\partial t} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \frac{\partial L}{\partial x} \cdot \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \frac{\partial L}{\partial u} \cdot \frac{\partial U(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \right)$$

$$\begin{aligned}
& +L \frac{d}{dt} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 - \frac{d}{dt} \frac{\partial F(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 - \lambda_j^i \frac{d}{dt} L \\
& + \psi(t) \cdot \left( \frac{\partial \varphi}{\partial t} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \frac{\partial \varphi}{\partial x} \cdot \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \frac{\partial \varphi}{\partial u} \cdot \frac{\partial U(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \right. \\
& \left. + \varphi \frac{d}{dt} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 - \frac{d}{dt} \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \right) \\
& = 0.
\end{aligned} \tag{12}$$

According to the maximality condition (4), the function

$$\psi_0 L(t, x(t), U(\alpha(t))) + \psi(t) \cdot \varphi(t, x(t), U(\alpha(t)))$$

attains an extremum for  $p(t) = \dot{p}(t) = \dots = p^{(m)}(t) = 0$ . Therefore

$$\psi_0 \frac{\partial L}{\partial u} \cdot \frac{\partial U(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \psi(t) \cdot \frac{\partial \varphi}{\partial u} \cdot \frac{\partial U(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 = 0$$

and (12) simplifies to

$$\begin{aligned}
& \psi_0 \left( \frac{\partial L}{\partial t} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \frac{\partial L}{\partial x} \cdot \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + L \frac{d}{dt} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \right. \\
& \left. - \frac{d}{dt} \frac{\partial F(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 - \lambda_j^i \frac{d}{dt} L \right) \\
& + \psi(t) \cdot \left( \frac{\partial \varphi}{\partial t} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \frac{\partial \varphi}{\partial x} \cdot \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \varphi \frac{d}{dt} \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \right. \\
& \left. - \frac{d}{dt} \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \right) \\
& = 0.
\end{aligned}$$

Using the adjoint system (3) and the property (5), one easily concludes that the above equality is equivalent to

$$\frac{d}{dt} \left( \psi_0 \frac{\partial F(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 + \psi_0 \lambda_j^i L + \psi(t) \cdot \frac{\partial X(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 - H \frac{\partial T(\alpha(t))}{\partial p_j^{(i)}} \Big|_0 \right) = 0.$$

## 4 Example

Consider the following simple time-optimal problem with  $n = r = 1$  and  $\Omega = (-1, 1)$ . Given two points  $\alpha$  and  $\beta$  in the state space  $R$ , we are to choose an admissible pair  $(x(\cdot), u(\cdot))$ , solution of the control equation

$$\dot{x}(t) = u(t),$$

and satisfying the boundary conditions  $x(0) = \alpha$ ,  $x(T) = \beta$ , in such a way that the time of transfer from  $\alpha$  to  $\beta$  is minimal:

$$T \rightarrow \min .$$

In this case the Lagrangian is given by  $L \equiv 1$  while  $\varphi = u$ . It is easy to conclude that the problem is invariant under the gauge symmetry

$$\begin{aligned} & g(t, x(t), u(t), p(t), \dot{p}(t), \ddot{p}(t)) \\ &= (p(t) + t, (\dot{p}(t) + 1)^2 x(t), 2\ddot{p}(t)x(t) + (\dot{p}(t) + 1)u(t)) , \end{aligned}$$

i.e., under

$$T = p(t) + t, \quad X = (\dot{p}(t) + 1)^2 x(t), \quad U = 2\ddot{p}(t)x(t) + (\dot{p}(t) + 1)u(t),$$

where  $p(\cdot)$  is an arbitrary function of class  $C^2([0, T]; R)$ . For that we choose  $F = p(t)$ ,  $\lambda^0 = \lambda^1 = \lambda^2 = 0$ , and conditions (8) and (9) follows:

$$L(T, X, U) \frac{d}{dt} T = \frac{d}{dt} (p(t) + t) = \frac{d}{dt} F + L(t, x(t), u(t)),$$

$$\begin{aligned} \varphi(T, X, U) \frac{d}{dt} T &= [2\ddot{p}(t)x(t) + (\dot{p}(t) + 1)u(t)] (\dot{p}(t) + 1) \\ &= \frac{d}{dt} [(\dot{p}(t) + 1)^2 x(t)] = \frac{d}{dt} X . \end{aligned}$$

From Theorem 1 the two non-trivial Noether currents

$$\psi_0 - H, \tag{13}$$

$$2\psi(t)x(t), \tag{14}$$

are obtained. As far as  $\psi_0$  is a constant, the Noether current (13) is just saying that the corresponding Hamiltonian  $H$  is constant along the Pontryagin extremals of the problem. This is indeed the case, since the problem under consideration is autonomous (cf. equality (5)). The Noether current (14) can be understood having in mind the maximality condition (4) ( $\frac{\partial H}{\partial u} = 0 \Leftrightarrow \psi(t) = 0$ ).

## 5 Concluding Remarks

In this paper we provide an extension of the second Noether's theorem to the optimal control framework. The result seems to be new even for the problems of the calculus of variations.

Theorem 1 admits several extensions. It was derived, as in the original work by Noether [13, 14], for state variables in an  $n$ -dimensional Euclidean space. It can be formulated, however, in contexts where the geometry is not Euclidean (these extensions can be found, in the classical context, e.g. in [8, 9, 15]). It admits also a generalization for optimal control problems which are invariant in a mixed sense, i.e, which are

invariant under a group of transformations depending upon  $\rho$  parameters and upon  $k$  arbitrary functions and their derivatives up to some given order. Other possibility is to obtain a more general version of the second Noether theorem for optimal control problems which does not admit exact symmetries. For example, under an invariance notion up to first-order terms in the functions  $p(\cdot)$  and its derivatives (cf. the quasi-invariance notion introduced by the author in [20] for the first Noether theorem). These and other questions, such as the generalization of the first and second Noether type theorems to constrained optimal control problems, are under study and will be addressed elsewhere.

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