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Inequalities Of Ostrowski Type And Applications In Numerical Integration *

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Abstract

Generalizations of Ostrowski integral inequality are derived. Applications in numerical integration are also given.

1 Introduction

In 1938, A. Ostrowski ([8] or [7, p. 468]) proved the following integral inequality:

THEOREM 1. Let $f : I \to \mathbf{R}$, where $I \subseteq \mathbf{R}$ is an interval, be a mapping differentiable in the interior I^o of I, and let $a, b \in I^o$ with $a < b$. If $|f'(x)| \leq M$ for all $x \in [a, b]$, then

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right] (b-a) M, \quad \forall x \in [a, b]. \tag{1}$$

In recent years a number of authors have generalized the above inequality, for example see [2], [3] and [9]. These generalizations can be applied to special means, in numerical integration (for quadrature formulas) etc.

In [1] the authors gave various estimates of the next 3-point quadrature rule:

$$R(f; a, b, x) = \int_a^b f(t)dt - \frac{1}{2} \left[(x-a)f(a) + (b-a)f(x) + (b-x)f(x) \right], \tag{2}$$

where $x \in [a, b]$ and $f : I \subset R \to R$ is a differentiable function in I^o, $a, b \in I^o$, $a < b$. They also suppose that $f' \in L_1(a, b)$ and $\gamma \leq f'(t) \leq \Gamma$, $t \in [a, b]$, where $\gamma, \Gamma \in R$ are constants. For example, in [1] we can find the following estimates

$$|R(f; a, b, x)| \leq \frac{\Gamma - \gamma}{4} \left[\frac{b-a}{2} + \left| x - \frac{a+b}{2} \right| \right] (b-a), \tag{3}$$

$$|R(f; a, b, x)| \leq \frac{\Gamma - \gamma}{4\sqrt{3}} (b-a) \left[\frac{(b-a)^2}{4} + 3 \left(x - \frac{a+b}{2} \right)^2 \right]^{1/2} \tag{4}$$

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and

$$|R(f;a,b,x)| \leq \frac{1}{4}\left[\frac{(b-a)^2}{4}+\left(x-\frac{a+b}{2}\right)^2\right]\left[\frac{\Gamma-\gamma}{2}+\left|S-\frac{\Gamma+\gamma}{2}\right|\right], \quad (5)$$

where $S=(f(b)-f(a))/(b-a)$. In [2] we can find the estimate:

$$|R(f;a,b,x)| \leq \frac{\Gamma-\gamma}{8}\left[(x-a)^2+(b-x)^2\right]. \quad (6)$$

In this paper we establish some new estimates of (2). We also use a completely new way of estimation (Theorem 3). Finally, we give applications in numerical integration.

2 Inequalities of Ostrowski Type

Let $g:[a,b]\to R$ be an absolutely continuous function. Let γ,Γ be real numbers such that $\gamma \leq g'(t) \leq \Gamma$, $t\in[a,b]$ $(a.e.)$. If $g'(t_0)$ does not exist, for some $t_0\in[a,b]$, then we set $g'(t_0)=(\Gamma+\gamma)/2$, by definition. This restriction does not affect the validity of the results obtained in this paper. That is, we consider such types of problems that the above restriction has no practical importance. It is only important from a theoretical point of view.

THEOREM 2. Let $f:[a,b]\to R$ be an absolutely continuous function. If there exist constants $\gamma,\Gamma\in R$ such that $\gamma\leq f'(t)\leq\Gamma$, $t\in[a,b]$ $(a.e.)$, then we have

$$\begin{aligned}&\left|\frac{1}{2}\left[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)\right]-\int_a^b f(t)dt\right|\\ \leq\ & \frac{S-\gamma}{2}(b-a)\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\end{aligned} \quad (7)$$

and

$$\begin{aligned}&\left|\frac{1}{2}\left[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)\right]-\int_a^b f(t)dt\right|\\ \leq\ & \frac{\Gamma-S}{2}(b-a)\left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right],\end{aligned} \quad (8)$$

where $S=(f(b)-f(a))/(b-a)$ and $x\in[a,b]$.

PROOF. Let us define the function

$$p(x,t)=\begin{cases} t-\frac{a+x}{2}, t\in[a,x]\\ t-\frac{x+b}{2}, t\in(x,b]\end{cases} \quad (9)$$

Integrating by parts, we have

$$\int_a^b p(x,t)f'(t)dt$$

$$
\begin{aligned}
&= \int_a^x \left(t - \frac{a+x}{2}\right) f'(t)dt + \int_x^b \left(t - \frac{x+b}{2}\right) f'(t)dt \\
&= \frac{1}{2}\left[(x-a)f(a) + (b-a)f(x) + (b-x)f(b)\right] - \int_a^b f(t)dt. \qquad (10)
\end{aligned}
$$

We also have

$$\int_a^b p(x,t)dt = \int_a^x \left(t - \frac{a+x}{2}\right) dt + \int_x^b \left(t - \frac{x+b}{2}\right) dt = 0. \qquad (11)$$

If $C \in R$ is a constant then we have

$$\int_a^b p(x,t)f'(t)dt = \int_a^b p(x,t)\left[f'(t) - C\right] dt. \qquad (12)$$

If we choose $C = \gamma$ in (12) then we have

$$\left|\int_a^b p(x,t)\left[f'(t) - \gamma\right] dt\right| \le \max|p(x,t)| \int_a^b |f'(t) - \gamma|\, dt. \qquad (13)$$

where max is taken for $t \in [a,b]$.

We also have

$$\max|p(x,t)| = \max\left\{\frac{x-a}{2}, \frac{b-x}{2}\right\} = \frac{1}{2}\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right] \qquad (14)$$

and

$$\int_a^b |f'(t) - \gamma|\, dt = f(b) - f(a) - \gamma(b-a) = (S-\gamma)(b-a). \qquad (15)$$

From (10), (12) and (13)-(15) we easily get (7). If we choose $C = \Gamma$ in (12), then we get

$$
\begin{aligned}
\left|\int_a^b p(x,t)\left[f'(t) - \Gamma\right] dt\right| &\le \max|p(x,t)| \int_a^b |f'(t) - \Gamma|\, dt \\
&= \frac{\Gamma - S}{2}(b-a)\left[\frac{b-a}{2} + \left|x - \frac{a+b}{2}\right|\right].
\end{aligned}
$$

This completes the proof.

REMARK 1. If we set $x = a$ or $x = b$ in the above theorem then we get corresponding trapezoid inequalities.

REMARK 2. If we choose $C = (\Gamma + \gamma)/2$ in (12) then we have

$$\left|\int_a^b p(x,t)\left[f'(t) - \frac{\Gamma+\gamma}{2}\right] dt\right| \le \max\left|f'(t) - \frac{\Gamma+\gamma}{2}\right| \int_a^b |p(x,t)|\, dt. \qquad (16)$$

From (16) and

$$\max\left|f'(t) - \frac{\Gamma+\gamma}{2}\right| \le \frac{\Gamma-\gamma}{2}, \quad t \in [a,b], \qquad (17)$$

$$\int_a^b |p(x,t)|\,dt = \frac{1}{2}\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)^2 \tag{18}$$

we get

$$\begin{aligned}&\left|\frac{1}{2}\left[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)\right]-\int_a^b f(t)dt\right|\\ \leq\quad&\frac{\Gamma-\gamma}{4}\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)^2,\end{aligned} \tag{19}$$

where $x \in [a,b]$. The same result is obtained in [2] under different conditions.

We now define the functional

$$\begin{aligned}S_\Psi(f,g) \quad=\quad & \int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt\int_a^b g(t)dt - \\ & -\frac{1}{\|\Psi\|_2^2}\int_a^b f(t)\Psi(t)dt\int_a^b g(t)\Psi(t)dt,\end{aligned}$$

where Ψ is a given square integrable function such that $\int_a^b \Psi(t)dt = 0,\ f,g \in L_2(a,b)$ and $\|\cdot\|_2$ is the usual norm in $L_2(a,b)$.

Some properties of such a functional are established in [10].

THEOREM 3. Let $f:[a,b]\to R$ be an absolutely continuous function such that $f' \in L_2(a,b)$. Then we have

$$\begin{aligned}&\left|\frac{1}{2}\left[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)\right]-\int_a^b f(t)dt\right|\\ \leq\quad&S_\Psi(f',f')^{1/2}\left\{\frac{b-a}{12}\left[\frac{(b-a)^2}{4}+3(x-\frac{a+b}{2})^2\right]\right\}^{1/2},\end{aligned} \tag{20}$$

where $x\in[a,b]$ and

$$\Psi(t) = \left\{\begin{array}{l} t-\frac{5a+x}{6},\, t\in\left[a,\frac{a+x}{2}\right] \\ t-\frac{a+5x}{6},\, t\in\left(\frac{a+x}{2},x\right] \\ t-\frac{5x+b}{6},\, t\in\left(x,\frac{x+b}{2}\right] \\ t-\frac{x+5b}{6},\, t\in\left(\frac{x+b}{2},b\right] \end{array}\right. . \tag{21}$$

Proof. Let $p(x,t)$ be defined by (9). Then we have

$$\begin{aligned}&\int_a^b p(x,t)\Psi(t)dt\\ =\quad&\int_a^{(a+x)/2}(t-\frac{a+x}{2})(t-\frac{5a+x}{6})dt+\int_{(a+x)/2}^{x}(t-\frac{a+x}{2})(t-\frac{a+5x}{6})dt\\ &+\int_x^{(x+b)/2}(t-\frac{x+b}{2})(t-\frac{5x+b}{6})dt+\int_{(x+b)/2}^{b}(t-\frac{x+b}{2})(t-\frac{x+5b}{6})dt\\ =\quad&0.\end{aligned} \tag{22}$$

From (10), (11) and (22) it follows

$$\begin{aligned} S_\Psi(p,f') &= \int_a^b p(x,t)f'(t)dt - \frac{1}{b-a}\int_a^b p(x,t)dt\int_a^b f'(t)dt \\ &\quad -\frac{1}{\|\Psi\|_2^2}\int_a^b p(x,t)\Psi(t)dt\int_a^b f'(t)\Psi(t)dt \\ &= \frac{1}{2}\left[(x-a)f(a)+(b-a)f(x)+(b-x)f(b)\right]-\int_a^b f(t)dt. \end{aligned} \tag{23}$$

On the other hand, we have ([10]),

$$|S_\Psi(p,f')| \le S_\Psi(p,p)^{1/2} S_\Psi(f',f')^{1/2}. \tag{24}$$

We also have

$$\begin{aligned} S_\Psi(p,p) &= \|p\|_2^2 - \frac{1}{b-a}\left(\int_a^b p(x,t)dt\right)^2 - \frac{1}{\|\Psi\|_2^2}\left(\int_a^b p(x,t)\Psi(t)dt\right)^2 \\ &= \frac{1}{12}\left[(x-a)^3+(b-x)^3\right] \\ &= \frac{b-a}{12}\left[\frac{(b-a)^2}{4}+3(x-\frac{a+b}{2})^2\right]. \end{aligned} \tag{25}$$

From (23)-(25) we easily get (20).

REMARK 3. Estimate (20) is better than (4). This is a consequence of the fact that $S_\Psi(g,g) \le (b-a)T(g,g)$, where $T(.,.)$ is the Chebyshev functional - see [10].

COROLLARY 1. Under the assumptions of Theorem 3 we have

$$\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)dt\right| \le S_\Psi(f',f')^{1/2}\frac{(b-a)^{3/2}}{2\sqrt{3}} \tag{26}$$

and

$$\left|\frac{f(a)+2f(\frac{a+b}{2})+f(b)}{4}(b-a)-\int_a^b f(t)dt\right| \le S_\Psi(f',f')^{1/2}\frac{(b-a)^{3/2}}{4\sqrt{3}}. \tag{27}$$

PROOF. We set $x=a$ and $x=(a+b)/2$ in (20) to get (26) and (27), respectively.

3 Applications in Numerical Integration

THEOREM 4. Let the assumptions of Theorem 2 hold. If $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ is a given subdivision of the interval $[a,b]$ then we have

$$\int_a^b f(t)dt = A(f,\xi,\pi)+R(f,\xi,\pi), \tag{28}$$

where

$$A(f,\xi,\pi)=\frac{1}{2}\sum_{i=0}^{n-1}\left[(\xi_i-x_i)f(x_i)+h_if(\xi_i)+(x_{i+1}-\xi_i)f(x_{i+1})\right] \tag{29}$$

and

$$|R(f,\xi,\pi)|\le\frac{\Gamma-\gamma}{4}\sum_{i=0}^{n-1}\left[\frac{1}{4}+\frac{(\xi_i-\frac{x_i+x_{i+1}}{2})^2}{h_i^2}\right]h_i^2, \tag{30}$$

for $h_i=x_{i+1}-x_i$, $x_i\le\xi_i\le x_{i+1}$, $i=0,1,2,...,n-1$.

PROOF. We apply (19) to the intervals $[x_i,x_{i+1}]$, $i=0,1,2,...,n-1$ and sum. Then the triangle inequality gives the proof.

COROLLARY 2. Under the assumptions of Theorem 4 we have

$$\int_a^b f(t)dt=A_T(f,\pi)+R_T(f,\pi), \tag{31}$$

where

$$A_T(f,\pi)=\frac{1}{2}\sum_{i=0}^{n-1}\left[f(x_i)+f(x_{i+1})\right]h_i \tag{32}$$

and

$$|R_T(f,\pi)|\le\frac{\Gamma-\gamma}{8}\sum_{i=0}^{n-1}h_i^2. \tag{33}$$

Indeed, we only need to apply Theorem 4 with $\xi_i=x_i$ for $i=0,1,2,...,n-1$.

COROLLARY 3. Under the assumptions of Theorem 4 we have

$$\int_a^b f(t)dt=A_S(f,\pi)+R_S(f,\pi), \tag{34}$$

where

$$A_S(f,\pi)=\frac{1}{4}\sum_{i=0}^{n-1}\left[f(x_i)+2f(\frac{x_i+x_{i+1}}{2})+f(x_{i+1})\right]h_i \tag{35}$$

and

$$|R_S(f,\pi)|\le\frac{\Gamma-\gamma}{16}\sum_{i=0}^{n-1}h_i^2. \tag{36}$$

Indeed, we only need to apply Theorem 4 with $\xi_i=(x_i+x_{i+1})/2$ for $i=0,1,2,...,n-1$.

In a similar way we can apply the rest of results obtained in Section 2 to establish some new approximations of integrals.

THEOREM 5. Let the assumptions of Theorem 4 hold. Then we have

$$\int_a^b f(t)dt=A(f,\xi,\pi)+R_Q(f,\xi,\pi), \tag{37}$$

where $A(f,\xi,\pi)$ is defined in Theorem 4,

$$|R_Q(f,\xi,\pi)| \le \sum_{i=0}^{n-1} \frac{S_i-\gamma}{2} h_i \left[\frac{h_i}{2} + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right] \tag{38}$$

or

$$|R_Q(f,\xi,\pi)| \le \sum_{i=0}^{n-1} \frac{\Gamma-S_i}{2} h_i \left[\frac{h_i}{2} + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right] \tag{39}$$

and $S_i = (f(x_{i+1}) - f(x_i))/h_i$ for $i = 0, 1, 2, ..., n-1$.

PROOF. We apply (7) to the interval $[x_i, x_{i+1}]$ to get

$$\begin{aligned} &\left|\frac{1}{2}\left[(\xi_i - x_i)f(x_i) + h_i f(\xi_i) + (x_{i+1}-\xi_i)f(x_{i+1})\right] - \int_{x_i}^{x_{i+1}} f(t)dt\right| \\ \le\ & \frac{S_i-\gamma}{2} h_i \left[\frac{h_i}{2} + \left|\xi_i - \frac{x_i+x_{i+1}}{2}\right|\right]. \end{aligned} \tag{40}$$

We also have

$$\begin{aligned} &\frac{1}{2}\left[(\xi_i - x_i)f(x_i) + h_i f(\xi_i) + (x_{i+1}-\xi_i)f(x_{i+1})\right] - \int_{x_i}^{x_{i+1}} f(t)dt \\ =\ & \int_{x_i}^{x_{i+1}} p(\xi_i,t)\left[f'(t)-\gamma\right]dt, \end{aligned}$$

where

$$p(\xi_i,t) = \begin{cases} t - \frac{x_i+\xi_i}{2}, t \in [x_i,\xi_i] \\ t - \frac{\xi_i+x_{i+1}}{2}, t \in (\xi_i, x_{i+1}] \end{cases} \tag{41}$$

If we sum the above relation over i from 0 to $n-1$ then we get (37). If we now apply the triangle inequality and (40) then we get (38).

In a similar way we can prove that (39) holds.

COROLLARY 4. Under the assumptions of Theorem 5 we have

$$\int_a^b f(t)dt = A_T(f,\pi) + R_T^S(f,\pi), \tag{42}$$

where $A_T(f,\pi)$ is defined in Corollary 2 and

$$\left|R_T^S(f,\pi)\right| \le \sum_{i=0}^{n-1} \frac{S_i-\gamma}{2} h_i^2 \tag{43}$$

or

$$\left|R_T^S(f,\pi)\right| \le \sum_{i=0}^{n-1} \frac{\Gamma-S_i}{2} h_i^2. \tag{44}$$

PROOF. We set $\xi_i = x_i$ in (37) to obtain (42) and set $\xi_i = x_i$ in (38) and (39) to obtain (43) and (44), respectively.

COROLLARY 5. Under the assumptions of Theorem 5 we have

$$\int_a^b f(t)dt = A_S(f,\pi) + R_S^Q(f,\pi), \tag{45}$$

where $A_S(f,\pi)$ is defined in Corollary 3 and

$$\left| R_S^Q(f,\pi) \right| \le \sum_{i=0}^{n-1} \frac{S_i - \gamma}{4} h_i^2 \tag{46}$$

or

$$\left| R_S^Q(f,\pi) \right| \le \sum_{i=0}^{n-1} \frac{\Gamma - S_i}{4} h_i^2. \tag{47}$$

PROOF. We set $\xi_i = (x_i + x_{i+1})/2$ in (37) to obtain (45) and set $\xi_i = (x_i + x_{i+1})/2$ in (38) and (39) to obtain (46) and (47), respectively.

Finally, let us compare the results obtained in this section. From the practical point of view, the most interesting results are contained in Corollary 3 and Corollary 5. Thus, we restrict our considerations to the mentioned corollaries.

In [3] we can find the following estimate

$$|R_S(f,\pi)| \le \frac{\|f'\|_\infty}{8} \sum_{i=0}^{n-1} h_i^2, \tag{48}$$

where $R_S(f,\pi)$ is defined in Corollary 3. If we choose $\gamma = \inf |f'(t)|$ and $\Gamma = \sup |f'(t)|$, $t \in [a,b]$ then $(\Gamma - \gamma)/16 \le \|f'\|_\infty /8$. Thus, the estimation (36) is better than the estimation (48). In fact, these two estimations are equal if and only if $\Gamma = -\gamma$. The last mentioned case is very rare in practice.

Hence, we have seen that the result obtained in Corollary 3 is generally better than the corresponding result obtained in [3]. On the other hand, the results obtained in Corollary 5 can be (much) better than the result of Corollary 3. We now give the following example to illustrate the last fact.

EXAMPLE 1. For the sake of simplicity we choose $x_i = a + ih$, $h = (b-a)/n$, $i = 0,1,2,...,n-1$, $x_n = b$. Let us consider the integral $\int_0^4 e^{t^2-16}dt$. We here have $f(t) = e^{t^2-16}$, $f'(t) = 2te^{t^2-16}$, $a = 0$, $b = 4$ such that $\gamma = 0$, $\Gamma = 8$, $f(a) = e^{-16}$, $f(b) = 1$. Then (36) becomes

$$|R_S(f,\pi)| \le \frac{\Gamma - \gamma}{16n}(b-a)^2 = \frac{8}{n} \tag{49}$$

and (46) becomes

$$\left| R_S^Q(f,\pi) \right| \le \frac{(b-a)^2}{4n} \left[\frac{f(b)-f(a)}{b-a} - \gamma \right] \le \frac{1}{n}. \tag{50}$$

It is obvious that (50) is better than (49). (Note that $R_S(f,\pi) = R_S^Q(f,\pi)$.)

REMARK 4. The results obtained in Corollary 5 are not always better than the result obtained in Corollary 3. For example, if we consider the monomial t^{λ}, $\lambda \in (1,4)$, $t \in [0,b]$ $(b>0)$, then we easily find that the estimate (36) is better than the estimate (46). Thus, both results have their fields of applicability.

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