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Asymptotic Behaviors Of Complex Analytic Dynamical Systems *

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Abstract

The main purpose of this paper is to prove that every analytic dynamical system on the complex plane has no limit cycle. Also we give analogues of the Denjoy-Wolff fixed point theorem of complex iteration, Schwarz Lemma of complex analyis and contraction principle in our settings.

The well known Poincare-Bendixson Theorem states that in a 2-dimensional smooth dynamical system, every bounded solution converges either to a limit cycle or to an equilibrium or the solution itself is periodic. The main purpose of this paper is to prove that an analytic system: $\dot{z} = f(z)$, where f is analytic on the complex plane C, does not have any limit cycle. Thus a bounded solution converges to an equilibrium or itself is periodic. We also give analogues of the Denjoy-Wolff fixed point theorem, Schwarz Lemma and contraction principle in the context of our analytic dynamical systems.

Consider the following system

$$\dot{z} = f(z) \tag{1}$$

where f is analytic on the complex plane C.

THEOREM 1. The system (1) does not have a limit cycle.

PROOF. Suppose to the contrary that the system (1) has a limit cycle $\gamma : [0, T] \to C$ with $\gamma(0) = \gamma(T)$. Let Ω be the region bounded by γ. Since Ω is bounded, there exists a semiflow: $\{\Phi_t \,|t \geq 0\}$ where $\Phi_t : \Omega \to \Omega$ is 1-1 and analytic, such that for each initial point $z \in \Omega$, the corresponding solution can be represented as $z(t) = \Phi_t(z)$ for all $t \geq 0$ [2, p.283]. Choose $n \in N$ (N is the set of positive integers) and let $h = T/n$. Then Φ_h is continuous on $\bar{\Omega}$ and analytic on Ω. Thus $\forall z \in \partial\Omega$, $\Phi_h(z) \in \partial\Omega$, and $\Phi_h^n(z) = \Phi_T(z) = z$. Let $g(z) = \Phi_h^n(z) - z$. Then g is continuous on $\bar{\Omega}$ and analytic on Ω. Since $|g(z)| = 0$ for all $z \in \partial\Omega$, by the Maximum module principle [3, p.134], $g(z) = 0$ for all $z \in \Omega$. Hence $\Phi_h^n(z) = z$ for all $z \in \Omega$. Thus we have shown that $\forall z \in \Omega$, $\Phi_h^n(z) = \Phi_T(z) = z$. So all solutions $z(t) = \Phi_t(z)$ in Ω are periodic. Therefore γ is not a limit cycle, which is a contradiction. The proof is complete.

COROLLARY 1. Let γ be a periodic solution for the system (1) such that $\gamma(0) = \gamma(T)$ and Ω the domain bounded by γ. Then (i) there exists a unique equilibrium $\xi \in \Omega$

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such that $f(\xi) = 0$ and $\mathrm{Re}(f'(\xi)) = 0$, and (ii) for each initial point $z \in \Omega\backslash\{\xi\}$, the corresponding solution $z(t)$ is periodic, $z(T) = z$ and encircles ξ.

PROOF. Since for every $z \in \gamma$, $f(z) \neq 0$ and lies in the tangent line of γ at z, the number of zeros of f in Ω counted with their multiplicities is equal to the winding number $W(f, r)$ of f with respect to γ, which satisfies [3,P.115]

$$1 = W(f, r) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz. \tag{2}$$

By the same argument in the proof of Theorem 1, we may prove that for each initial point $z \in \Omega\backslash\{\xi\}$, the corresponding solution $z(t)$ satisfies $z(T) = z$ where T is the period of γ. Suppose the unique equilibrium ξ satisfies $\mathrm{Re}(f'(\xi)) < 0$. Then by Lyapunov's stability Theorem [4], there exists $N_s(\xi)$, a neighborhood of ξ, such that $\forall z \in N_s(\xi)$, $z(t) \to \xi$ as $t \to \infty$. So ξ is a local attractor. Hence every solution near ξ is certainly not periodic which contradicts the fact shown above. Similarly, if $\mathrm{Re}(f'(\xi)) > 0$, then there exists $N_w(\xi)$, a neighborhood of ξ, such that $\forall z \in N_w(\xi)$, $\exists t_1 > 0$ such that $\Phi_{t_1} \notin N_w(\xi)$. Thus from the above cases we have shown that every solution near ξ is certainly not periodic which contradicts the fact shown above. So we have $\mathrm{Re}(f'(\xi)) = 0$.

Next, let γ_1 be an arbitrary periodic solution in Ω. By the same argument as above, there exists $\xi_1 \in \Omega_1$ (the domain bounded by γ_1) such that $f(\xi_1) = 0$. By (2), $\xi_1 = \xi$. In view of the facts just shown, we have proved that every solution $z(t)$ in Ω with initial point $z \notin \xi$ is periodic and encircles ξ. The proof is complete.

The following is an analogue of Denjoy-Wolff fixed point Theorem for (1).

THEOREM 2. Suppose the bounded and simply connected domain Ω in C is invariant under the system (1) (i.e.,for all $z \in \Omega$, the corresponding solution $z(t) \in \Omega$ for all $t \geq 0$). If $\forall z \in \Omega$, $f(z) \neq 0$, then $\exists \xi \in \partial\Omega$ such that every solution $z(t) \to \xi$ as $t \to \infty$ once $z(0) \in \Omega$.

PROOF. Since Ω is a bounded and simply connected domain, by Riemann's mapping theorem [3, p.230], there exists a conformal mapping $M : \Omega \to \Delta$ such that $M(\partial\Omega) \subset \partial\Delta$ (where $\partial\Omega$ is the boundary of Ω, while Δ and $\partial\Delta$ denote the open unit disc and its boundary respectively). Since Ω is bounded and invariant there exists a semiflow $\{\Phi_t \,|\forall t \geq 0\}$ where $\Phi_t : \Omega \to \Omega$ is 1-1 and anaylytic.. Choose $h > 0$ which is sufficiently small. Then Φ_h is analytic on Ω. Let $F = M \circ \Phi_h \circ M^{-1}$. Then $F : \Delta \to \Delta$ is analytic. By the Theorem of normal family [3, p.224], the set $\{F^k \,|\, k = 1, 2, 3, ...\}$ of iterates of F contains a convergent subsequence $F^{k_i} \to G$ as $i \to \infty$, where G is analytic on Δ and the convergence is uniform for each compact set contained in Δ. Since $f(z) \neq 0$ for all $z \in \Omega$, by Theorem 1, it follows that every solution $z(t) = \Phi_t(z)$ converges to a point $\alpha \in \partial\Omega$ as $t \to \infty$. This implies that $F^k(w)$ tends to a point in $\partial\Delta$ as $k \to \infty$. Hence the mapping G maps Δ to $\partial\Delta$. So we have $|G(w)| = 1$ for all $w \in \Delta$. In turn, we have the following chain of implications:

$$\frac{\partial}{\partial w}(\overline{G(w)}G(w)) = 0 \Rightarrow \frac{\partial \bar{G}}{\partial w} G + \bar{G}\frac{\partial G}{\partial w} = 0 \Rightarrow \frac{\partial G}{\partial w} = 0,$$

since $\frac{\partial \bar{G}}{\partial w} = 0$ and since G is analytic. Thus $G(w) \equiv \zeta$ for some $\zeta \in \partial\Delta$. Thus

$$\forall w \quad \in \quad \Delta,\ F^k(w) \to \zeta \ \text{ as } \ k \to \infty,$$

$$\begin{aligned}&\Rightarrow\quad \forall z\in\Omega,\ \Phi_h^k(z)\to\zeta,\ \xi=M^{-1}(\zeta)\ \text{ as }\ k\to\infty,\\ &\Rightarrow\quad \forall z\in\Omega,\ z(t)=\Phi_t(z)\to\zeta\ \text{ as }\ t\to\infty.\end{aligned}$$

The proof is complete.

The following result can be regarded as an analogue of Schwarz Lemma [3, p.135] for (1).

THEOREM 3. Assume $f(0)=0$ and $\forall b\in\partial\Delta$, $\mathrm{Re}(f(b)\bar{b})<0$. Then $\exists K>0$ and $\delta>0$ such that (i) every solution $z(t)$ of (1) with initial point $z\in\Delta$ satisfies $|z(t)|\le Ke^{-\delta t}|z|$, and (ii) $\mathrm{Re}(f'(0))<0$.

PROOF. Clearly, the condition $\mathrm{Re}(f(b)\bar{b})<0$ implies that $\forall b\in\partial\Delta$, $f(b)\ne 0$ and $f(b)$ points toward the interior of Δ. So Δ is invariant under the system (1). Then there exists a semiflow $\{\Phi_t|t\ge 0\}$ where$\Phi_t:\Delta\to\Delta$ is 1-1 and analytic such that every solution $z(t)$ satisfying $z(0)=z$ can be expressed as $z(t)=\Phi_t(z)$, and $z=\Phi_0(z)$. Since $\partial\Delta$ is compact, $\exists h>0$ sufficiently small such that. $\Phi_h(b)\in\Delta$ for all $b\in\partial\Delta$. Let $F=\Phi_h$. Then F is analytic on Δ and continuous on $\bar{\Delta}$. Since $f(0)=0$, $F(0)=\Phi_h(0)=0$. Let $G(z)=F(z)/z$. Then G is analytic on Δ, and

$$\max_{|b|=1}|G(b)|=\max_{|b|=1}|F(b)/b|=\alpha<1.$$

By the Maximum Module Principle,

$$|F(z)|\le\alpha|z|\Rightarrow|F^k(z)|\le\alpha^k|z|=e^{kln\alpha}|z|,\ \forall k=1,2,3,\dots\ .$$

For each $t\ge 0$, $t=kh+r$, where $k\in N$ and $0\le r<h$. It follows that

$$\Phi_t(z)=\Phi_{kh+r}(z)=\Phi_h^k(\Phi_r(z)),\tag{3}$$

and

$$|\Phi_t(z)|\le e^{-kh\delta}|z|=e^{-(kh+r)\delta}e^{r\delta}|z|\le Ke^{-\delta t}|z|,\tag{4}$$

where $\delta=-\ln(\alpha)/h$ and $K=1/\alpha$. Hence (i) is proved.

Since the convergence $F^k(z)\to 0$ as $k\to\infty$ is uniform for each compact set in Δ, by Cauchy integral formula [3, p.114], we have

$$\begin{aligned}\lim_{k\to\infty}(F^k)'(0) &= \lim_{k\to\infty}\frac{1}{2\pi i}\int_{C_r}\frac{F^k(z)}{z^2}dz\ \ (C_r=\{z\,|\,|z|=r,z\in C\}),\\ &= \frac{1}{2\pi i}\int_{C_r}\lim_{k\to\infty}\frac{F^k(z)}{z^2}dz\\ &= \frac{1}{2\pi i}\int_{C_r}0dz\\ &= 0.\end{aligned}$$

Thus

$$\lim_{k\to\infty}(\Phi_h^k)(0)'=\lim_{k\to\infty}(\Phi_h'(0))^k=0\Rightarrow|\Phi_h'(0)|<1\Rightarrow\mathrm{Re}(f'(0))<0\tag{5}$$

since h is sufficiently small. This completes the proof.

The following result can be regarded as an analogue of the contraction principle.

THEOREM 4. Let Ω be a bounded, simply connected and invariant domain for the system (1). Suppose $\forall b \in \partial\Omega$, $f(b)$ is not 0 and points toward the interior of Ω. Then (i) $\exists \xi \in \Omega$ such that . $\forall z \in \Omega$, the corresponding solution $z(t) \to \xi$ as $t \to \infty$, and (ii) $\mathrm{Re}(f'(\xi)) < 0$.

PROOF. From the assumption that $\forall b \in \partial\Omega$, $f(b)$ points toward the interior of Ω, it follows that each solution $z(t)$ with $b \in \partial\Omega$ as the initial point is forced to flow into Ω, i.e., $z(t) \in \Omega$ for every $t > 0$. Since Ω is invariant, the semiflow $\{\Phi_t \,|t \geq 0\}$, where $\Phi_t : \Omega \to \Omega$ is 1-1 and analytic, exists such that the solution $z(t)$ can be represented as $z(t) = \Phi_t(z)$, if $z(0) = z \in \Omega$. Choose $h > 0$ sufficiently small. Then $\Phi_h : \bar{\Omega} \to \Omega$ is continuous and analytic on Ω. By Riemann mapping theorem, there exists conformal mapping $M : \Omega \to \Delta$, since Ω is bounded and simply connected. Then $G = M \circ \Phi_h \circ M^{-1} : \Delta \to \Delta$ is analytic, and G is continuous on $\bar{\Delta}$. By Brouwer fixed point theorem [5] and the assumption of the Theorem, $\exists \zeta \in \Delta$ such that $G(\zeta) = \zeta$. Let $Q(z) = (z - \xi)/(1 - \bar{\xi} z)$ where $\xi = M^{-1}(\zeta)$. Then $Q(\xi) = 0$. Set $H = QGQ^{-1}$. Then $H(0) = 0$ and H is analytic on Δ. By the argument in the proof of Theorem 3(i), it follows that $\exists K > 0$ and $\delta > 0$ such that .

$$|Q \circ M \circ \Phi_h \circ M^{-1} \circ Q^{-1}(z)| \leq K e^{-\delta z} |z|.$$

Hence $\Phi_t(z) \to \xi$ as $t \to \infty$, where $\xi = M^{-1}(\zeta)$. The proof of $\mathrm{Re}(f'(\xi)) < 0$ is similar to that of Theorem 3(ii).

EXAMPLE. Consider the equation

$$\dot{z} = i(z^2 - 1).$$

Let $f(z) = i(z^2 - 1)$. Then 1 and -1 are equilibria and $f'(1) = 2i$, $f'(-1) = -2i$. By a direct derivation, we obtain that for each solution $z(t)$ if $z(0) \notin i\Re \cup \{\pm 1\}$, then $z(t)$ has to satisfy

$$z(t) = \frac{1 + \frac{z(0)-1}{z(0)+1} + e^{i2t}}{1 - \frac{z(0)-1}{z(0)+1} + e^{i2t}}$$

Hence $z(t)$ is periodic with period π. Note that the imaginary axis $i\Re = \{iy \,|\, y \in \Re\}$ is invariant, and the system can be reduced to $\dot{y} = -(y^2 + 1)$. Hence the solution $y(t) = \tan(\tan^{-1}(y(0)) - t)$ tends to $-\infty$ as $t \to \tan^{-1}(y(0)) + \pi/2$.

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