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Linear Differential Polynomials Sharing Three Values With Weights *

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Abstract

We prove some uniqueness theorems for meromorphic functions that share weighted values.

1 Introduction and Definitions

Let f and g be two nonconstant meromorphic functions defined in the open complex plane $\mathbf{C}$. If for some $a \in \mathbf{C} \cup \{\infty\}$ the roots of $f-a$ and $g-a$ (if $a=\infty$, roots of $f-a$ and $g-a$ are the poles of f and g respectively) coincide in locations and multiplicities we say that f and g share the value a CM (counting multiplicities) and if coincide in locations only we say that f and g share a IM (ignoring multiplicities). We do not explain the standard notations of the value distribution theory because those are available in [3]. However, we explain some definitions which will be needed in the sequel. Also we denote by f, g two nonconstant meromorphic functions defined on $\mathbf{C}$ unless otherwise stated.

DEFINITION 1 ([6]). If s is a positive integer, we denote by $N(r,a;f \mid = s)$ the counting function of those a-points of f whose multiplicity is s, where we count an a-point according to its multiplicity.

DEFINITION 2 ([6]). If s is a positive integer, we denote by $\overline{N}(r,a;f \mid \geq s)$ the counting function of those a-points of f whose multiplicities are greater than or equal to s, where each a-point is counted only once.

DEFINITION 3 (cf. [1, 6]). If s is a positive integer, we denote by $N_s(r,a;f)$ the counting function of a-points of f where an a-point of multiplicity m is counted m times if $m \leq s$ and s times if $m > s$. We put $N_\infty(r,a;f) = N(r,a;f)$.

DEFINITION 4 ([6]). Let f, g share a value a IM. We denote by $\overline{N}_*(r,a;f,g)$ the counting function of those a-points of f whose multiplicities are not equal to multiplicities of the corresponding a-points of g, where each such a-point is counted only once.

Clearly $\overline{N}_*(r,a;f,g) \equiv \overline{N}_*(r,a;g,f)$.

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DEFINITION 5 (cf. [2]). We put

$$T_0(r,f)=\int_1^r \frac{T(t,f)}{t}dt,\ N_0(r,a;f)=\int_1^r \frac{N(t,a;f)}{t}dt,$$

$$N_s^0(r,a;f)=\int_1^r \frac{N_s(t,a;f)}{t}dt,\ m_0(r,f)=\int_1^r \frac{m(t,f)}{t}dt,$$

$$m_0(r,a;f)=\int_1^r \frac{m(t,a;f)}{t}dt,\ S_0(r,f)=\int_1^r \frac{S(t,f)}{t}dt.$$

DEFINITION 6 (cf. [11]). We define $\delta_s(a;f)$ as $\delta_s(a;f)=1-\limsup\limits_{r\to\infty}\frac{N_s(r,a;f)}{T(r,f)}$ where $a\in C\cup\{\infty\}$.

Clearly

$$0\le\delta(a;f)\le\delta_s(a;f)\le\delta_{s-1}(a;f)\le\cdots\le\delta_2(a;f)\le\delta_1(a;f)=\Theta(a;f)\le 1,$$

where $\Theta(a;f)=1-\limsup\limits_{r\to\infty}\frac{\overline{N}(r,a;f)}{T(r,f)}$.

DEFINITION 7 (cf. [2, 5]). We put $\delta_0(a;f)=1-\limsup\limits_{r\to\infty}\frac{N_0(r,a;f)}{T_0(r,f)}$, $\Theta_0(a;f)=1-\limsup\limits_{r\to\infty}\frac{\overline{N}_0(r,a;f)}{T_0(r,f)}$, $\delta_s^0(a;f)=1-\limsup\limits_{r\to\infty}\frac{N_s^0(r,a;f)}{T_0(r,f)}$ where $a\in C\cup\{\infty\}$.

Yang in [9] asked: what can be said about the relationship between two entire functions f and g if f,g share 0 CM and f',g' share 1 CM?

To answer this question, Yi [10] proved the following theorem.

THEOREM A. Let f and g be two nonconstant entire functions. Assume that f,g share 0 CM and $f^{(n)},g^{(n)}$ share 1 CM and $2\delta(0;f)>1$, where n is a nonnegative integer. Then either $f^{(n)}\cdot g^{(n)}\equiv 1$ or $f\equiv g$.

Inspired by this result, in [4], the following question was asked: what can be said about the relationship between two meromorphic functions f,g when two differential polynomials, generated by them, share certain values?

Let $\psi(D)=\sum_{i=1}^{p}\alpha_i D^i$ be a linear differential operator with constant coefficients where $D\equiv d/dz$ (cf. [4]). The following theorem was proved in [4].

THEOREM B ([4]). Let f, g be of finite order such that f,g share ∞ CM, $\psi(D)f,\psi(D)g$ are nonconstant and share $0,1$ CM, and $\frac{\sum_{a\neq\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))}-\frac{3(1-\Theta(\infty;f))}{2\sum_{a\neq\infty}\delta(a;f)}>\frac{1}{2}$, where $\sum_{a\neq\infty}\delta(a;f)>0$. Then either (a) $[\psi(D)f][\psi(D)g]\equiv 1$ or (b) $f-g\equiv q$, where $q=q(z)$ is a solution of the differential equation $\psi(D)w=0$. Further, if f has at least one pole or $\psi(D)f$ has at least one zero, then the possibility (a) does not arise.

The purpose of the paper is to make a twofold improvement of *Theorem B:* firstly by weakening the condition on deficiencies and secondly by relaxing the nature of sharing of values. In order to relax the nature of sharing values we consider a gradation of sharing of values which measures how close a shared value is to being shared IM or to being shared CM and is called weight of the sharing.

DEFINITION 8 ([6, 7]). Let k be a nonnegative integer or infinity. For $a\in\mathbf{C}\cup\{\infty\}$ we denote by $E_k(a;f)$ the set of all a-points of f where an a-point of multiplicity m

is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k then z_0 is a zero of $f - a$ with multiplicity m $(\leq k)$ if and only if it is a zero of $g - a$ with multiplicity m $(\leq k)$ and z_0 is a zero of $f - a$ with multiplicity m $(> k)$ if and only if it is a zero of $g - a$ with multiplicity n $(> k)$ where m is not necessarily equal to n.

We say that f, g share (a, k) if f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

2 Lemmas

In this section we present some lemmas which are necessary in the sequel.

LEMMA 1. If f, g share $(0, 0), (1, 0)$ and $(\infty, 0)$, then (i) $T(r, f) \leq 3T(r, g) + S(r, f)$, and (ii) $T(r, g) \leq 3T(r, f) + S(r, g)$.

PROOF. The lemma follows from the second fundamental theorem (see p.43 in [3]).

LEMMA 2. Let $c_1 f + c_2 g \equiv c_3$, where c_1, c_2 and c_3 are nonzero constants. Then (i) $T(r, f) \leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + S(r, f)$, and (ii) $T(r, g) \leq \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; g) + S(r, g)$.

PROOF. By the second fundamental theorem [3], we get

$$\begin{aligned} T(r, f) &\leq \overline{N}(r, 0; f) + \overline{N}(r, c_3/c_1; f) + \overline{N}(r, \infty; f) + S(r, f) \\ &= \overline{N}(r, 0; f) + \overline{N}(r, 0; g) + \overline{N}(r, \infty; f) + S(r, f), \end{aligned}$$

which is (i). In a similar way we can prove (ii). This proves the lemma.

LEMMA 3. Let f, g share $(1, 0)$ and $h = f'/f - g'/g$. If $\overline{N}(r, 1; f) \neq S(r, f)$ and $h \equiv 0$, then $f \equiv g$.

PROOF. Since $h \equiv 0$, it follows that $f \equiv cg$, where c is a constant. Since f, g share $(1, 0)$ and $\overline{N}(r, 1; f) \neq S(r, f)$, there exists $z_0 \in \mathbf{C}$ such that $f(z_0) = g(z_0) = 1$ so that $c = 1$. Therefore $f \equiv g$. This completes the proof.

LEMMA 4. If f, g share $(0, 0), (1, 1)$ and $(\infty, 0)$ and $f \not\equiv g$, then (i) $\overline{N}(r, 1; f \mid\geq 2) \leq \overline{N}_*(r, 0; f, g) + \overline{N}_*(r, \infty; f, g) + S(r, f)$, and (ii) $\overline{N}(r, 1; g \mid\geq 2) \leq \overline{N}_*(r, 0; f, g) + \overline{N}_*(r, \infty; f, g) + S(r, f)$.

PROOF. Since $\overline{N}(r, 1; f) \equiv \overline{N}(r, 1; g)$, the lemma is obvious if $\overline{N}(r, 1; f) = S(r, f)$. So we suppose that $\overline{N}(r, 1; f) \neq S(r, f)$. Let $h = f'/f - g'/g$. Since $f \not\equiv g$, by Lemma 3 we get $h \not\equiv 0$. Also since f, g share $(1, 1)$, a multiple 1-point of f is a multiple 1-point of g and vise-versa and so it is a zero of f' and g'. Hence

$$\overline{N}(r, 1; f \mid\geq 2) \leq N(r, 0; h) \leq T(r, f) + O(1) = N(r, h) + m(r, h) + O(1),$$

i.e.,

$$\overline{N}(r, 1; f \mid\geq 2) \leq N(r, h) + S(r, f), \tag{1}$$

by Milloux theorem (see p.55 in [3]) and Lemma 1.

The possible poles of h occur at the zeros and poles of f, g. Clearly if z_0 is a zero or a pole of f and g with the same multiplicity then z_0 is not a pole of h. Since all the poles of h are simple, it follows that

$$N(r,h) = \overline{N}(r,h) \le \overline{N}_*(r,0;f,g) + \overline{N}_*(r,\infty;f,g). \tag{2}$$

Now (i) follows from (1) and (2). Also (ii) follows from (i) because f, g share $(1,1)$ so that $\overline{N}(r,1;g \mid\ge 2) \equiv \overline{N}(r,1;f \mid\ge 2)$. This completes our proof.

LEMMA 5. If f, g share $(1,1)$ and $H \not\equiv 0$, where $H = \frac{f''}{f'} - \frac{2f'}{f-1} - \frac{g''}{g'} + \frac{2g'}{g-1}$, then (i) $N(r,1;f \mid= 1) \le N(r,H) + S(r,f) + S(r,g)$, and (ii) $N(r,1;g \mid= 1) \le N(r,H) + S(r,f) + S(r,g)$.

PROOF. Since f, g share $(1,1)$, it follows that a simple 1-point of f is a simple 1-point of g and conversely. Let z_0 be a simple 1-point of f and g. Then in some neighbourhood of z_0 we get

$$f - 1 = (z - z_0)\alpha \text{ and } g - 1 = (z - z_0)\beta$$

where α, β are analytic at z_0 and $\alpha(z_0) \ne 0$, $\beta(z_0) \ne 0$. This implies by a simple calculation that in some neighbourhood of z_0

$$H = (z - z_0)\left[\frac{\alpha\alpha'' - 2(\alpha')^2}{\alpha\{\alpha + (z - z_0)\alpha'\}} - \frac{\beta\beta'' - 2(\beta')^2}{\beta\{\beta + (z - z_0)\beta'\}}\right].$$

This shows that z_0 is a zero of H. Hence

$$\begin{aligned} N(r,0;f|{=}1) &\le N(r,0;H) \le T(r,H) + O(1) = N(r,H) + m(r,H) + O(1) \\ &= N(r,H) + S(r,f) + S(r,g) \end{aligned}$$

by Milloux theorem (see p.55 in [3]).

Now (ii) follows from (i) because $N(r,1;g \mid= 1) = N(r,1;f \mid= 1)$. The proof is complete.

LEMMA 6. Let f, g share $(0,0), (1,0)$ and $(\infty,0)$ and $H \not\equiv 0$, where H is defined as in Lemma 5. Then

$$\begin{aligned} N(r,H) \le\ & \overline{N}_*(r,0;f,g) + \overline{N}_*(r,\infty;f,g) + \overline{N}_*(r,1;f,g) \\ & + N_\otimes(r,0;f') + N_\otimes(r,0;g'), \end{aligned}$$

where $N_\otimes(r,0;f')$ is the counting function of those zeros of f' which are not the zeros of f and $f-1$ and $N_\otimes(r,0;g')$ is the analogous quantity.

PROOF. The possible poles of H occur at (i) multiple zeros of f, g; (ii) zeros of $f-1$, $g-1$; (iii) poles of f, g; and (iv) zeros of f', g' which are not the zeros of f, $f-1$ and g, $g-1$ respectively. Let z_0 be a zero of $f-1$ and $g-1$ with multiplicities m and n respectively. Then in some neighbourhood of z_0, we get $f - 1 = (z - z_0)^m\alpha$ and $g - 1 = (z - z_0)^n\beta$, where α, β are analytic at z_0 and $\alpha(z_0) \ne 0$, $\beta(z_0) \ne 0$. Then in some neighbourhood of z_0 we get

$$H(z) = \frac{m-n}{z-z_0}\phi(z) + \psi(z),$$

where ϕ, ψ are analytic at z_0 and $\phi(z_0) \neq 0$. This shows that if $m = n$, then $z_0$is not a pole of H and if $m \neq n$ then z_0 is a simple pole of H.

Similarly we can show that if z_1 is a zero or a pole of f, g with multiplicities m and n respectively, then z_1 is not a pole of H if $m = n$ and z_1 is a simple pole of H if $m \neq n$.

Since all poles of H are simple, the lemma follows from the above discussion. The proof is complete.

LEMMA 7 ([2]) $\lim_{r\to\infty} \frac{S_0(r,f)}{T_0(r,f)} = 0$ for all values of r.

LEMMA 8 (cf. [5, 8]). For $a \in C \cup \{\infty\}$, $\delta(a;f) \leq \delta_0(a;f)$, $\Theta(a;f) \leq \Theta_0(a;f)$ and $\delta_s(a;f) \leq \delta_s^0(a;f)$.

LEMMA 9 ([5]) (i) $\limsup_{r\to\infty} \frac{T_0(r,\psi(D)f)}{T_0(r,f)} \geq \sum_{a\neq\infty} \delta_p^0(a;f)$, and (ii) $\delta_0(0;\psi(D)f) \geq \frac{\sum_{a\neq\infty}\delta_0(a;f)}{1+p(1-\Theta(\infty;f))}$.

LEMMA 10 ([5]). If $\sum_{a\neq\infty} \delta_p^0(a;f) > 0$ then $\Theta_0(\infty;\psi(D)f) \geq 1 - \frac{1-\Theta(\infty;f)}{\sum_{a\neq\infty}\delta_p^0(a;f)}$.

3 Theorems

In this section we present the main results of the paper.

THEOREM 1. Let $\psi(D)f$, $\psi(D)g$ be nonconstant such that (i) f, g share $(\infty, 0)$; (ii) $\psi(D)f$, $\psi(D)g$ share $(0,1), (1,1)$; and (iii) $\frac{\sum_{a\neq\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))} > \frac{1}{2} + \frac{2(1-\Theta(\infty;f))}{\sum_{a\neq\infty}\delta_p(a;f)}$, where $\sum_{a\neq\infty}\delta_p(a;f) > 0$. Then either $[\psi(D)f][\psi(D)g] \equiv 1$ or $f - g \equiv q$, where $q = q(z)$ is a solution of the differential equation $\psi(D)w = 0$.

THEOREM 2. Let $\psi(D)f$, $\psi(D)g$ be nonconstant such that (i) f, g share (∞, ∞); (ii) $\psi(D)f$, $\psi(D)g$ share $(0,1), (1,1)$; and (iii) $\frac{\sum_{a\neq\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))} > \frac{1}{2} + \frac{1-\Theta(\infty;f)}{\sum_{a\neq\infty}\delta_p(a;f)}$, where $\sum_{a\neq\infty}\delta_p(a;f) > 0$. Then either $[\psi(D)f][\psi(D)g] \equiv 1$ or $f - g \equiv q$, where $q = q(z)$ is a solution of the differential equation $\psi(D)w = 0$.

REMARK 1. If f has at least one pole or $\psi(D)f$ has at least one zero then the possibility $[\psi(D)f][\psi(D)g] \equiv 1$ does not arise in Theorems 1 and 2.

The following example shows that the theorems are sharp.

EXAMPLE 1. Let $f = \frac{-1}{4}\exp(z) + \frac{1}{6}\exp(2z)$, $g = \frac{1}{6}\exp(-z) - \frac{1}{14}\exp(-2z)$ and $\psi(D) = D^2 - 5D$. Then $\psi(D)f$, $\psi(D)g$ share $(0,\infty), (1,\infty)$ and f, g share (∞,∞). Also $\sum_{a\neq\infty}\delta(a;f) = 1/2$ and $\Theta(\infty;f) = 1$ but neither $[\psi(D)f][\psi(D)g] \equiv 1$ nor $f - g \equiv c_1 + c_2\exp(5z)$ for any constants c_1, c_2.

We shall prove Theorem 1 only because Theorem 2 can be proved similarly noting that $\overline{N}_*(r,\infty;f,g) \equiv 0$ when f, g share (∞,∞).

Proof of Theorem 1. Let $F = \psi(D)f$ and $G = \psi(D)g$. Then clearly F, G share $(0,1)$, $(1,1)$, $(\infty,0)$ and in view of Lemma 8 , Lemma 9 and Lemma 10, the given condition implies $2\delta_2^o(0;F) + 4\Theta_o(\infty;F) > 5$.

Let $F \not\equiv G$. We shall show that $F.G \equiv 1$. If possible, suppose that $H \not\equiv 0$, where $H = \frac{F''}{F'} - \frac{2F'}{F-1} - \frac{G''}{G'} + \frac{2G'}{G-1}$. Now by the second fundamental theorem (see p.43 in [3]) and Lemma 1 we get

$$\begin{aligned} T(r,f) + T(r,g) &\le \overline{N}(r,0;F) + \overline{N}(r,1;F) + \overline{N}(r,\infty;F) \\ &\quad + \overline{N}(r,0;G) + \overline{N}(r,1;G) + \overline{N}(r,\infty;G) \\ &\quad - N_{\otimes}(r,0;F') - N_{\otimes}(r,0;G') + S(r,F). \end{aligned}$$

Since F, G share $(0,1), (1,1), (\infty,0)$, we obtain

$$\begin{aligned} T(r,f) + T(r,g) &\le 2\overline{N}(r,0;F) + 2\overline{N}(r,1;G) + 2\overline{N}(r,\infty;F) \\ &\quad - N_{\otimes}(r,0;F') - N_{\otimes}(r,0;G') + S(r,f). \end{aligned} \tag{3}$$

Again since F, G share $(1,1)$, we get

$$\begin{aligned} 2\overline{N}(r,1;G) &\le N(r,1;G \mid= 1) + N(r,1;G) \\ &\le N(r,1;F \mid= 1) + T(r,G) + O(1). \end{aligned}$$

So from (3) we get

$$\begin{aligned} T(r,F) &\le 2\overline{N}(r,0;F) + N(r,1;F \mid= 1) + 2\overline{N}(r,\infty;F) \\ &\quad - N_{\otimes}(r,0;F') - N_{\otimes}(r,0;G') + S(r,F). \end{aligned} \tag{4}$$

Since F, G share $(0,1), (1,1)$, it follows that $\overline{N}_*(r,0;F,G) \le \overline{N}(r,0;F \mid\ge 2)$ and $\overline{N}_*(r,1;F,G) \le \overline{N}(r,1;F \mid\ge 2)$ and so by Lemma 1, Lemma 4, Lemma 5 and Lemma 6, we get

$$\begin{aligned} N(r,1;F|{=}1) &\le N(r,H) + S(r,F) \\ &\le \overline{N}_*(r,0;F,G) + \overline{N}_*(r,1;F,G) + \overline{N}_*(r,\infty;F,G) \\ &\quad + N_{\otimes}(r,0;F') + N_{\otimes}(r,0;G') + S(r,F) \\ &\le \overline{N}(r,0;F \mid\ge 2) + \overline{N}(r,1;F \mid\ge 2) + \overline{N}(r,\infty;F) \\ &\quad + N_{\otimes}(r,0;F') + N_{\otimes}(r,0;G') + S(r,F) \\ &\le 2\overline{N}(r,0;F \mid\ge 2) + 2\overline{N}(r,\infty;F) \\ &\quad + N_{\otimes}(r,0;F') + N_{\otimes}(r,0;G') + S(r,F). \end{aligned}$$

So from (4) we obtain

$$\begin{aligned} T(r,F) \le 2\overline{N}(r,0;F) + 2\overline{N}(r,0;F| \ge 2) + 4\overline{N}(r,\infty;F) + S(r,F) \\ +2N_2(r,0;F) + 4\overline{N}(r,\infty : F) + S(r,F), \end{aligned}$$

which gives on integration

$$T_0(r,F) \le 2N_2^0(r,0;F) + 4\overline{N}_0(r,\infty;F) + S_0(r,F). \tag{5}$$

Henceforth ϵ stands for a quantity satisfying

$$0 < 2\epsilon < 2\delta_2^0(0;F) + 4\Theta_0(\infty;F) - 5.$$

Now from (5) we get by Lemma 7

$$\begin{aligned} T_0(r,F) &< \{6-2\delta_2^0(0;F)-4\Theta_0(\infty;F)+\epsilon+o(1)\}T_0(r,F) \\ &< \{1-\epsilon+o(1)\}T_0(r,F), \end{aligned}$$

which is a contradiction. Therefore $H \equiv 0$ and so

$$\frac{1}{F-1} \equiv \frac{A}{G-1} + B, \tag{6}$$

where A and B are constants. Since F is nonconstant, $A \neq 0$. Let $B = 0$. Since $F \not\equiv G$, it follows that $A \neq 1$. Then we get from (6) $F + \frac{G}{A} \equiv 1 - \frac{1}{A}$. So by Lemma 2 and Lemma 7 we obtain on integration

$$\begin{aligned} T_0(r,F) &\leq \overline{N}_0(r,0;F)+\overline{N}_0(r,0;G)+\overline{N}_0(r,\infty;F)+S_0(r,F) \\ &\leq 2N_2^0(r,0;F)+4\overline{N}_0(r,\infty;F)+S_0(r,F) \\ &< \{6-2\delta_2^0(0;F)-4\Theta_0(\infty;F)+\epsilon+o(1)\}T_0(r,F) \\ &< \{1-\epsilon+o(1)\}T_0(r,F), \end{aligned}$$

which is a contradiction. So $B \neq 0$. Let $A \neq B$. If $B = -1$, from (6) we get $\frac{A-B-1}{F} - BG \equiv A - B$. Since G is nonconstant , $A - B - 1 \neq 0$ and so by Lemma 2, Lemma 7 and the first fundamental theorem we get on integration

$$T_0(r,\frac{1}{F}) \leq \overline{N}_0(r,0;\frac{1}{F}) + \overline{N}_0(r,0;G) + \overline{N}_0(r,\infty;\frac{1}{F}) + S_0(r,F)$$

i.e.

$$\begin{aligned} T_0(r,F) &\leq 2N_2^0(r,0;F)+4\overline{N}_0(r,\infty;F)+S_0(r,F) \\ &< \{6-2\delta_2^0(0;F)-4\Theta_0(\infty;F)+\epsilon+o(1)\}T_0(r,F) \\ &< \{1-\epsilon+o(1)\}T_0(r,F), \end{aligned}$$

which is a contradiction. So $B \neq -1$ and hence from (6) we get

$$\frac{BF}{1+B} - \frac{\frac{A-B}{B} - \frac{A-B-1}{1+B}}{G + \frac{A-B}{B}} \equiv 1.$$

Clearly $\frac{A-B}{B} - \frac{A-B-1}{B+1} \neq 0$ and so by Lemma 2 and Lemma 7 we get on integration

$$\begin{aligned} T_0(r,F) &\leq \overline{N}_0(r,0;F)+\overline{N}_0(r,0;\frac{1}{G+\frac{A-B}{B}})+\overline{N}_0(f,\infty;F)+S_0(r,F) \\ &\leq 2N_2^0(r,0:F)+4\overline{N}_0(R,\infty;F)+S_0(r,F) \\ &< \{6-2\delta_2^0(0;F)-4\Theta_0(\infty;F)+\epsilon+o(1)\}T_0(r,F) \\ &< \{1-\epsilon+o(1)\}T_0(r,F), \end{aligned}$$

which is a contradiction. So $A = B$ and hence from (6) we get

$$F + \frac{1}{BG} = \frac{1+B}{B}.$$

If $B \neq -1$, we get by Lemma 2 and Lemma 7 on integration

$$\begin{aligned} T_0(r,F) &\leq \overline{N}_0(r,0;F) + \overline{N}_0(r,0;\frac{1}{G}) + \overline{N}_0(r,\infty;F) + S_0(r,F) \\ &\leq 2N_2^0(r,0;F) + 4\overline{N}_0(r,\infty;F) + S_0(r,F) \\ &< \{6 - 2\delta_2^0(0;F) - 4\Theta(\infty;F) + \epsilon + o(1)\}T_0(r,F) \\ &< \{1 - \epsilon + o(1)\}T_0(r,F), \end{aligned}$$

which is a contradiction. Hence $A = B = -1$and so from (6) we get $F.G \equiv 1$. Therefore either $F.G \equiv 1$ or $F \equiv G$ and so either $[\psi(D)f][\psi(D)g] \equiv 1$ or $f - g \equiv q$, where $q = q(z)$ is a solution of the differential equation $\psi(D)w = 0$. This proves our theorem.

As an application of Theorem 1 we get the following corollary.

COROLLARY 1. Suppose (i) $f^{(p)}$, $g^{(p)}$are nonconstant and share $(0,1), (1,1), (\infty,0)$; (ii) $\frac{\sum_{a\neq\infty}\delta(a;f)}{1+p(1-\Theta(\infty;f))} > \frac{1}{2} + \frac{2(1-\Theta(\infty;f))}{\sum_{a\neq\infty}\delta_p(a;f)}$, where $\sum_{a\neq\infty}\delta_p(a;f) > 0$; and (iii) $\Theta(0;f) + \Theta(0;g) + \Theta(\infty;f) > 2$. Then either $f \equiv g$ or $f^{(p)}.g^{(p)} \equiv 1$. Further, if $f^{(p)}$ has at least one zero or pole, the possibility $f^{(p)} \cdot g^{(p)} \equiv 1$ does not arise.

PROOF. By Theorem 1 we see that either $f^{(p)}.g^{(p)} \equiv 1$ or $f - g \equiv q$ where q is a polynomial. Since $\sum_{a\neq\infty}\delta_p(a;f) > 0$, it follows that f is transcendental. If $q \not\equiv 0$, by Nevanlinna's three small functions theorem [3], we get

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,q;f) + \overline{N}(r,\infty;f) + S(r,f) \\ &= \overline{N}(r,0;f) + \overline{N}(r,0;g) + \overline{N}(r,\infty;f) + S(r,f), \end{aligned}$$

which implies $\Theta(0;f) + \Theta(0;g) + \Theta(\infty;f) \leq 2$ because $T(r,g) = \{1 + o(1)\}T(r,f)$. This contradiction shows that $q \equiv 0$ and so $f \equiv g$. This proves the corollary.

Let us conclude the paper with the following question: Is it possible to relax the sharing $(0,1), (1,1)$ to the sharing $(0,0), (1,0)$ in Theorems 1 and 2?

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