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Positive and Bounded Solutions of Discrete Reaction-Diffusion Equations *

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Abstract

By means of the technique of separation of variables, the problem of the existence of positive and bounded solutions of two discrete reaction-diffusion equations are reduced to that of ordinary difference equations. Monotone and comparison methods are then employed to construct positive and bounded solutions of the latter equations.

Discretization of the heat equation lead to several well known multilevel partial difference schemes (such as the DuFort-Frankel scheme [1, p.302]). These schemes, or equations, can also be obtained by first principles. Consider, for example, the distribution of heat through a "long" rod. Assume that the rod is so long that it can be labeled by the set of integers. Let $u_m^{(n)}$ be the temperature at the integral position m and integral time n of the rod. At time n, if the temperature $u_{m-1}^{(n)}$ is higher than $u_m^{(n)}$, heat will flow from the point $m-1$ to m. The amount of increase is $u_m^{(n+1)} - u_m^{(n)}$, and it is reasonable to postulate that the increase is proportional to the quantity $u_{m-1}^{(n)} - u_m^{(n)}$, say, $r\left(u_{m-1}^{(n)} - u_m^{(n)}\right)$ where r is a positive constant. By symmetry considerations, it is then reasonable that the total effect is

$$u_m^{(n+1)} - u_m^{(n)} = r\left(u_{m-1}^{(n)} - u_m^{(n)}\right) + r\left(u_{m+1}^{(n)} - u_m^{(n)}\right).$$

Such a postulate can be regarded as a discrete Newton law of cooling.

If this equation represents a real model, it will be reasonable to expect that it has a positive bounded solution for appropriate initial temperature distributions. Indeed, the double sequence $\left\{u_m^{(n)}\right\} \equiv \{1\}$ is such a solution.

The question then arises as to whether a more general nonhomogeneous rod subject to delayed feedback has a positive and bounded solution. In this note, we will consider two such cases the first of which is modeled by an equation of the form

$$u_m^{(n+1)} - u_m^{(n)} = \alpha u_{m-1}^{(n)} + \beta u_m^{(n)} + \gamma u_{m+1}^{(n)} + q u_m^{(n-\sigma)}, \tag{1}$$

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where $n = 0, 1, 2, ...$ and $m = 0, \pm 1, \pm 2, ...$ and σ is a nonnegative integer. We will develop several techniques which will enable us to find sufficient conditions for the existence of positive and bounded solutions. For related references, [2,3,4] can be consulted.

First of all, for the sake of convenience, we will denote the set of nonnegative integers by N and the set of integers by Z. Given an arbitrary set of initial values $u_m^{(n)}$, $-\sigma \le n \le 0, m \in Z$, we can successively calculate $u_0^{(1)}; u_{-1}^{(1)}, u_0^{(2)}, u_1^{(1)}; u_{-2}^{(1)}, u_{-1}^{(2)}, u_0^{(3)}, u_1^{(2)}, u_2^{(1)}; ...$ in a unique manner. Such a double sequence $u = \{u_m^{(n)} | m \in Z, n = -\sigma, -\sigma + 1, ...\}$ is called a solution of (1). Suppose there is some nonnegative integer T such that $u_m^{(n)} > 0$ for $m \in Z$ and $n \ge T$, then u is said to be eventually positive.

By designating the doubly infinite sequence $\left\{..., u_{-1}^{(n)}, u_0^{(n)}, u_1^{(n)}, ...\right\}$ as the *column* vector $u^{(n)}$, we see that a solution of (1) can also be regarded as a vector sequence $\{u^{(n)}\}_{n=-\sigma}^{\infty}$. Furthermore, such a sequence satisfies the delay vector recurrence relation

$$u^{(n+1)} - u^{(n)} = Au^{(n)} + qu^{(n-\sigma)},\ n \in N, \tag{2}$$

where $A = (a_{ij})$ is an infinite matrix with diagonal elements all equal to β, subdiagonal elements all equal to α, and superdiagonal elements to γ, that is, $a_{ii} = \beta, a_{i,i-1} = \alpha$ and $a_{i,i+1} = \gamma$ for $i \in Z$, and zero elsewhere.

Let Ω be the linear space of all real doubly infinite sequences over the real field and under the usual operations. If we call a vector v in Ω positive (denoted by $v > 0$) when all its components are positive, then clearly a solution u of (1) is eventually positive if, and only if the vector sequence $\{u^{(n)}\}$ is eventually positive. Therefore, (1) has an eventually positive solution if, and only if, the relation (2) has an eventually positive solution. Next, we observe that if there is a number λ and a corresponding vector v such that

$$Av = \lambda v, \tag{3}$$

then for any solution $\{x_n\}_{n=-\sigma}^{\infty}$ of the scalar difference equation

$$x_{n+1} - x_n = \lambda x_n + qx_{n-\sigma},\ n \in N, \tag{4}$$

we have

$$x_{n+1}v - x_n v = \lambda x_n v + qx_{n-\sigma}v = x_n Av + qx_{n-\sigma}v,\ n \in N,$$

that is, $\{x_n v\}$ is a solution of (2).

In view of these, in order to find an eventually positive and bounded solution of (2), it suffices to find a number λ and a corresponding positive and bounded vector v such that (3) is satisfied, as well as an eventually positive and bounded solution of (4).

Clearly, when $\lambda = \alpha + \beta + \gamma$, (3) is satisfied by the positive and bounded constant sequence $v = \{1\}$. Now, the corresponding scalar difference equation is

$$x_{n+1} - x_n = (\alpha + \beta + \gamma)x_n + qx_{n-\sigma},\ n \in Z. \tag{5}$$

To find an eventually positive and bounded solution of (5), we may look for one that is of the form $\{t^n\}$ and $0 < t \le 1$. Substituting the unknown solution into (5), we obtain the following characteristic equation

$$t^{\sigma+1} - (1 + \alpha + \beta + \gamma)t^{\sigma} - q = 0.$$

Therefore, it suffices to find a root of this equation in $(0, 1]$. This is a relatively easy problem. Indeed, consider the polynomial

$$h(t) = t^{\sigma+1} + at^{\sigma} + b,\ t \in R. \tag{6}$$

When $\sigma = 0$, the unique root is $t = -a - b$. Hence it has a root in $(0, 1]$ if, and only if, $-1 \leq a + b < 0$. Suppose now $\sigma > 0$. Note that $h(0) = b$ and $\lim_{t\to\infty} h(t) = \infty$, we see that (6) has a positive root when $b < \dot{0}$. In such a case, it has a root in $(0, 1]$ if, and only if, $h(1) = 1 + a + b \geq 0$. If $b = 0$, then (6) has the unique roots 0 and $-a$, hence it has a root in $(0, 1]$ if, and only if, $-1 \leq a < 0$. If $b > 0$ and $a \geq 0$, then $h(t) > 0$ for $t > 0$, so that $h(t)$ does not have any positive roots. Finally, when $b > 0$ and $a < 0$, the function $h(t)$ has a minimum at $t^* = -a\sigma/(\sigma + 1)$, and

$$h(t^*) = b - \frac{(-a)^{\sigma+1}\sigma^{\sigma}}{(\sigma + 1)^{\sigma+1}}.$$

Hence $h(t)$ has a root in $(0, 1]$ if, and only if,

$$b \leq \frac{(-a)^{\sigma+1}\sigma^{\sigma}}{(\sigma + 1)^{\sigma+1}}$$

and

$$0 < t^* = \frac{-a\sigma}{\sigma + 1} \leq 1.$$

In particular, when $\sigma = 0$, (5) has a solution of the form $\{t^n\}$ where $t \in (0, 1]$ if, and only if,

$$0 \geq \alpha + \beta + \gamma + q > -1;$$

and when $\sigma \geq 1$, (5) has a solution of the form $\{t^n\}$ where $t \in (0, 1]$ if, and only if,

$$0 < \alpha + \beta + \gamma \leq \frac{1}{\sigma} \text{ and } -\frac{(1 + \alpha + \beta + \gamma)^{\sigma+1}\sigma}{(\sigma + 1)^{\sigma+1}} \leq q < 0, \tag{7}$$

or,

$$-1 < \alpha + \beta + \gamma \leq 0 \text{ and } -\frac{(1 + \alpha + \beta + \gamma)^{\sigma+1}\sigma^{\sigma}}{(\sigma + 1)^{\sigma+1}} \leq q \leq -(\alpha + \beta + \gamma), \tag{8}$$

or,

$$\alpha + \beta + \gamma \leq -1 \text{ and } 0 < q \leq -(\alpha + \beta + \gamma). \tag{9}$$

THEOREM 1. When $\sigma = 0$, if $-1 < \alpha + \beta + \gamma + q \leq 0$, then (1) has an eventually positive and bounded solution. When $\sigma \geq 1$, if (7) or (8) or (9) holds, then (1) has an eventually positive and bounded solution.

We remark that if an eventually positive and zero convergent solution $\{u^{(n)}\}$ of (2) is desired, we need only to modify the conditions in the above Theorem so that (5) has a solution of the form $\{t^n\}$ where $t \in (0, 1)$ instead of $(0, 1]$.

We now consider another case where the coefficients α, β, γ are dependent on the variable m and the coefficient q is time dependent so that we are now dealing with the equation

$$u_m^{(n+1)} - u_m^{(n)} = \alpha_m u_{m-1}^{(n)} + \beta_m u_m^{(n)} + \gamma_m u_{m+1}^{(n)} + q_n u_m^{(n-\sigma)}, m \in Z, n \in N. \tag{10}$$

By modifying the above procedures, we may easily come up with the following result.

THEOREM 2. Suppose there is a real number λ such that the steady state equation

$$\alpha_m v_{m-1} + \beta_m v_m + \gamma_m v_{m+1} = \lambda v_m,\ m \in Z, \tag{11}$$

has a positive bounded solution $\{v_m\}_{m=-\infty}^{\infty}$, and suppose further that the scalar difference equation

$$x_{n+1} - x_n = \lambda x_n + q_n x_{n-\sigma},\ n \in N \tag{12}$$

has an eventually positive and bounded solution, then (10) has an eventually positive and bounded solution.

In view of Theorem 2, there are two things we need to take care. First, let us try to find a positive constant solution $\{c\}$ of (11). This leads to $\lambda = \alpha_m + \beta_m + \gamma_m$. In other words, when $\lambda = \alpha_m + \beta_m + \gamma_m$ for $m \in Z$, then (11) has a positive and bounded solution.

THEOREM 3. Suppose $\lambda = \alpha_m + \beta_m + \gamma_m$ for all $m \in Z$. Then (11) has positive and bounded solutions of the form $\{c\}$.

Next, we need to find conditions which are sufficient for the existence of an eventually positive and bounded solution of (12). First, note that when $\sigma = 0$, equation (12) becomes

$$x_{n+1} = (1 + \lambda + q_n)x_n,\ n \in N,$$

which clearly has an eventually positive and bounded solution when $-1 < \lambda + q_n \leq 0$ for $n \in N$.

Next, assume that $\sigma > 0$. Note that if $q_n \leq q$ for $n \in N$, and (4) has an eventually positive and bounded solution $\{x_n\}$ such that $x_{n-\sigma} > 0$ for $n \geq M$, then letting $\{y_n\}$ be the solution of (12) determined by the conditions $y_{M-\sigma} = x_{M-\sigma}, ..., y_M = x_M$, we see that

$$x_{M+1} - y_{M+1} = (1 + \lambda)(x_M - y_M) + qx_{M-\sigma} - q_n y_{M-\sigma} \geq 0.$$

By induction, it is easily seen that $x_n \geq y_n$ for $n > M$. Next, note that

$$y_{n+1} = (1 + \lambda)y_n + q_n y_{n-\sigma}.$$

If we impose the conditions $1 + \lambda \geq 0$ and $q_n > 0$ for $n \in N$, or, $1 + \lambda > 0$ and $q_n \geq 0$ for $n \in N$, then clearly $y_n > 0$ for $n > M$. We have thus shown the following.

THEOREM 4. Suppose $\sigma > 0$. Suppose either $1 + \lambda \geq 0$ and $0 < q_n \leq q$ for $n \in N$, or, $1 + \lambda > 0$ and $0 \leq q_n \leq q$ for $n \in N$. If (4) has an eventually positive and bounded solution, then (12) has an eventually positive and bounded solution.

The set of sufficient conditions described above is not the only one possible. Indeed, we assert that equation (12) will have an eventually positive and bounded solution provided that $\lambda \leq 0$, $q \leq q_n < 0$ for $n \in N$ and (4) has an eventually positive and bounded solution. To this end, let $x = \{x_n\}$ be an eventually positive and bounded solution of (4). By summing (4), we see that

$$x_n \geq -\sum_{i=n}^{\infty} \lambda x_i - \sum_{i=n}^{\infty} q_i x_{i-\sigma} \geq 0$$

for n larger than or equal to some positive integer M. Define the mapping T on the set of all real sequences of the form $y = \{y_n\}_{n=-\sigma}^{\infty}$ by

$$(Ty)_n = -\sum_{i=n}^{\infty} \lambda y_i - \sum_{i=n}^{\infty} q_i y_{i-\sigma},\ n \geq M,$$

and

$$(Ty)_n = y_M,\ -\sigma \leq n < M$$

Consider the following sequence $\{y^{[t]}\}$ of successive approximations defined by

$$y_n^{[0]} = \begin{cases} x_n & n \geq M \\ x_M & -\sigma \leq n < M \end{cases}, \tag{13}$$

and

$$y^{[t+1]} = Ty^{[t]},\ t = 0, 1, 2, \dots .$$

Clearly,

$$y_n^{[1]} = \left(Ty^{[0]}\right)_n = (Tx)_n = -\sum_{i=n}^{\infty} \lambda x_i - \sum_{i=n}^{\infty} q_i x_{i-\sigma},\ n \geq M,$$

so that

$$0 \leq y_n^{[1]} \leq x_n, n \geq M.$$

Thus,

$$0 \leq y_n^{[2]} = \left(Ty^{[1]}\right)_n \leq (Tx)_n = y_n^{[1]},\ n \geq M.$$

By induction, we see that

$$0 \leq \dots \leq y_n^{[t+1]} \leq y_n^{[t]} \leq \dots \leq y_n^{[0]} = x_n,\ n \geq M.$$

Thus as $t \to \infty$, $y^{[t]}$ converges pointwise to some nonnegative sequence w which satisfies

$$w_n = -\sum_{i=n}^{\infty} \lambda w_i - \sum_{i=n}^{\infty} q_i w_{i-\sigma} \tag{14}$$

for $n \geq M$ and $w_n = x_M > 0$ for $-\sigma \leq n < M$. By taking differences on both sides of the above equation, we see that w is an eventually nonnegative and bounded solution of (12). Finally, we show that w is eventually positive. Indeed, suppose to the contrary that there exists an integer $n^* > M \geq 1$ such that $w_n > 0$ for $-\sigma \leq n < n^*$ and $w_{n^*} = 0$. Then in view of (14), we see that

$$0 = w_{n^*} = -\sum_{i=n^*}^{\infty} \lambda w_i - \sum_{i=n^*}^{\infty} q_i w_{i-\sigma}.$$

Since $q_n < 0$ for $n \geq n^*$, thus $w_{n^*-\sigma} = 0$, which is a contradiction. We have thus shown the following.

THEOREM 5. If $\sigma > 0, \lambda \leq 0,\ q \leq q_n < 0$ for $n \in N$ and (4) has an eventually positive and bounded solution, then (12) has an eventually positive and bounded solution.

We remark that in the above Theorem, the assumption that $q_n < 0$ for $n \in N$ can be relaxed to $q_n \leq 0$ for $n \in N$ and $q_i \neq 0$ for all $i \in [n-\sigma, n]$ where $n \in N$.

THEOREM 6. If $\sigma > 0, 1+\lambda < 0, 0 < 1/q_n \leq 1/q$ for $n \in N$ and (4) has an eventually positive and bounded solution, then (12) has an eventually positive and bounded solution.

The proof is similar to that of Theorem 5, but there are some technically different details. Let $\{x_n\}$ be an eventually positive solution of (4) so that

$$x_n = \frac{1}{q}\left\{x_{n+\sigma+1} - (1+\lambda)x_{n+\sigma}\right\} \geq 0$$

for n larger than or equal to some positive integer M. Let H be the mapping defined on the set of all real sequences of the form $y = \{y_n\}_{n=-\sigma}^{\infty}$ as follows:

$$(Hy)_n = \frac{1}{q_{n+\sigma}}\left\{y_{n+\sigma+1} - (1+\lambda)y_{n+\sigma}\right\},\ n \geq M,$$

and

$$(Hy)_n = y_M,\ -\sigma \leq n < M.$$

Consider the following sequence $\{y^{[t]}\}$ of successive approximations defined by

$$y_n^{[0]} = \begin{cases} x_n & n \geq M \\ x_M & -\sigma \leq n < M \end{cases},$$

and

$$y^{[t+1]} = Hy^{[t]},\ t = 0, 1, 2, \ldots.$$

Clearly,

$$\left(Hy^{[0]}\right)_n = (Hx)_n = \frac{1}{q_{n+\sigma}}\left\{x_{n+\sigma+1} - (1+\lambda)x_{\bar{n}\sigma}\right\},\ n \geq M,$$

thus

$$0 \leq y_n^{[1]} \leq x_n,\ n \geq M.$$

By induction, we see that

$$0 \leq \ldots \leq y_n^{[t+1]} \leq y_n^{[t]} \leq \ldots \leq y_n^{[0]} = x_n,\ n \geq M.$$

Thus as $t \to \infty$, $y^{[t]}$ converges pointwise to some nonnegative sequence w which satisfies

$$w_n = \frac{1}{q_{n+\sigma}}\left\{w_{n+\sigma+1} - (1+\lambda)w_{n+\sigma}\right\} \tag{15}$$

for $n \geq M$ and $w_n = x_M > 0$ for $-\sigma \leq n < M$. By taking differences on both sides of the above equation, we see that w is an eventually nonnegative and bounded solution of (12). Finally, we show that w is eventually positive. Indeed, suppose to the contrary that there exists an integer $n^* > M \geq 1$ such that $w_n > 0$ for $-\sigma \leq n < n^*$ and $w_{n^*} = 0$. Then in view of (15), we see that

$$0 \leq w_{n^*+\sigma+1} = (1+\lambda)w_{n^*+\sigma} \leq 0,$$

so that $w_{n^*+\sigma} = w_{n^*+\sigma+1} = 0$. But then the same reasoning leads to

$$w_{n^*+2\sigma} = w_{n^*+2\sigma+1} = w_{n^*+2\sigma+2} = 0.$$

By induction, it is then not difficult to see that there is some integer Γ such that $w_n = 0$ for $n \geq \Gamma$. But then in view of

$$w_{n^*-1} = \frac{1}{q_{n^*+\sigma-1}} \left\{ w_{n^*+\sigma} - (1+\lambda)w_{n^*+\sigma-1} \right\},$$

$$w_{n^*+\sigma} = \frac{1}{q_{n^*+2\sigma}} \left\{ w_{n^*+2\sigma+1} - (1+\lambda)w_{n^*+2\sigma} \right\},$$

$$w_{n^*+\sigma+1} = \frac{1}{q_{n^*+2\sigma+1}} \left\{ w_{n^*+2\sigma+2} - (1+\lambda)w_{n^*+2\sigma+1} \right\}, \ \ldots,$$

and the fact that there is some positive integer t such that $n^* + t\sigma \geq \Gamma$, we see that $w_{n^*-1} = 0$. This is a contradiction.

By combining the above results in different ways, we may obtain several sets of sufficient conditions for the existence of eventually positive and bounded solutions of (10). For instance, suppose the following conditions hold:

(1) $\alpha_m + \beta_m + \gamma_m = \alpha_0 + \beta_0 + \gamma_0$ for $m \in Z$,
(2) $\sigma > 0$,
(3) $-1 < \alpha_0 + \beta_0 + \gamma_0 \leq 0$, and
(4) $0 \leq q_n \leq -(\alpha_0 + \beta_0 + \gamma_0)$ for $n \in N$.

Then (10) has an eventually positive and bounded solution.

References

[1] G. E. Forsythe and W. R. Wasow, Finite Difference Methods for Partial Difference Methods, 2^{nd} ed., Clarendon Press, Oxford, 1978.

[2] S. S. Cheng and R. Medina (1999), Growth conditions for a discrete heat equation with delayed control, Dynamic Sys. Appl., 8, 361-367.

[3] S. S. Cheng and J. Y. Lin (1998), Stability criteria for a discrete reaction-diffusion equation, Far East J. Math. Sci., 6(3), 425-435.

[4] S. S. Cheng and G. H. Lin (1998), Green's function and stability of a partial difference scheme, Comput. Math. Appl., 35(5), 27-41.