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Periodic Solutions Of Second Order Integro-differential Equations *

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Abstract

We consider second order integro-differential systems with periodic boundary conditions. Assuming that the upper solution is less than the lower solution, the existence of periodic solution by the method of mixed monotony is obtained.

1 Introduction

In this paper we shall consider periodic boundary value problem (PBVP) of second order integro-differential system of the form

$$-u_i'' = f_i\left(t, u, T^i u\right),\ 0 < t < 2\pi, i = 1, 2, ..., N, \tag{1}$$

with boundary condition

$$u(0) = u(2\pi),\ u'(0) = u'(2\pi), \tag{2}$$

where $f \in C\left(I \times R^N \times R^N, R^N\right)$, $I = [0, 2\pi]$, and $T^i = \left(T^{i1}, ..., T^{iN}\right)$, $1 \le i \le N$, are some bounded integral operators which are nondecraesing. For example,

$$\left(T^{ij}u\right)(t) = \int_0^t a^{ij}(s)\, u_j(s)\, ds, t \in I$$

where $a^{ij}(s)$ are nonnegative functions on I.

There are many results for the periodic boundary value problem without integral terms, see [2] and the references therein. The method of upper and lower solutions is widely used to discuss the existence, uniqueness, boundedness, stability and asymptotic behavior of the solutions. Usually it is assumed that the upper solution is larger than the lower solution when we use monotone method. The question is whether the existence of solution holds by reversing the order of the upper and lower solutions. Some results are given in [2]. On the other hand, the method of mixed monotony has been used to discuss the existence of boundary value problems by [1], [3] and [4] under

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the assumption that the upper solution is greater than the lower solution. In this note we shall consider the reverse order case and discuss the existence of the solution of periodic BVP by the method of mixed monotony. This will complement the result of [1] and generalize the result in section 5.3 of [2]. For convenience, we use the notation:

$$[\beta, \alpha] = \left\{u \in C\left(I, R^N\right) \mid \beta \leq u \leq \alpha \text{ on } I\right\}$$

here $\beta \leq u$ means that $\beta_i \leq u_i$ for $1 \leq i \leq N$.

THEOREM 1 ([1]). Assume that there exist a positive constant ε, a function $F \in C\left(I \times R^N \times R^N \times R^N \times R^N, R^N\right)$, and two functions $\alpha, \beta \in C^2\left(I, R^N\right)$ such that the following conditions hold :

(A1) $\alpha(t) \leq \beta(t)$, $t \in I$.

(A2) for all $u, v \in C\left(I, R^N\right)$ with $\alpha \leq u \leq \beta$ and $\alpha \leq v \leq \beta$, we have

$$\begin{gathered}\alpha_i'' + F_i\left(t, u, T^i u, v, T^i v\right) \geq -\tfrac{1}{2\varepsilon}\left[(v_i - \alpha_i) + (u_i - \alpha_i)\right] \text{ on } I, \\ \alpha(0) = \alpha(2\pi), \; \alpha'(0) \geq \alpha'(2\pi).\end{gathered}$$

(A3) for all $u, v \in C\left(I, R^N\right)$ with $\alpha \leq u \leq \beta$ and $\alpha \leq v \leq \beta$, we have

$$\begin{gathered}\beta_i'' + F_i\left(t, u, T^i u, v, T^i v\right) \leq -\tfrac{1}{2\varepsilon}\left[(v_i - \beta_i) + (u_i - \beta_i)\right] \text{ on } I, \\ \beta(0) = \beta(2\pi), \; \beta'(0) \leq \beta'(2\pi).\end{gathered}$$

(A4) $F_i(t, u, y, v, z)$ is nondecreasing in u and y and nonincreasing in v and z respectively for fixed the remaining arguments.

(A5) $F_i\left(t, u, T^i u, u, T^i u\right) = f_i\left(t, u, T^i u\right)$, $1 \leq i \leq N$.

(A6) If there exist two functions $\rho, \gamma \in C^2\left(I, R^N\right)$ such that

$$\begin{gathered}\rho_i'' + F_i\left(t, \rho, T^i \rho, \gamma, T^i \gamma\right) = -\tfrac{1}{2\varepsilon}\left(\gamma_i - \rho_i\right) \text{ on } I, \\ \rho(0) = \rho(2\pi), \; \rho'(0) = \rho'(2\pi),\end{gathered}$$

and

$$\begin{gathered}\gamma_i'' + F_i\left(t, \gamma, T^i \gamma, \rho, T^i \rho\right) = -\tfrac{1}{2\varepsilon}\left(\rho_i - \gamma_i\right) \text{ on } I, \\ \gamma(0) = \gamma(2\pi), \; \gamma'(0) = \gamma'(2\pi),\end{gathered}$$

then $\rho \equiv \gamma$ on I.

Then the problem (1)-(2) has a unique solution u with $\alpha(t) \leq u(t) \leq \beta(t)$ on I.

LEMMA 2 ([2, Lemma 5.11]). Assume that $g \in C(I)$, $g \geq 0$ on I and g is not trivial. Then any solution to the boundary value problem

$$\begin{gathered}u'' + ku = g(t) \\ u(0) = u(2\pi), \; u'(0) = u'(2\pi) + \lambda\end{gathered}$$

is positive on I for any $\lambda \geq 0$ if and only if $0 < k \leq 1/4$.

2 Main Results

We shall consider the existence of the solution for periodic boundary value problem (1)-(2) under the assumption that the upper solution is less than the lower solution. The method of mixed monotony will be used.

THEOREM 3. Assume that there exist two functions $\alpha, \beta \in C^2\left(I, R^N\right)$ and a function $F \in C\left(I \times R^N \times R^N \times R^N \times R^N, R^N\right)$ such that the following conditions hold :

(B1) $\beta(t) \leq \alpha(t)$ on I.

(B2) For all $u, v \in C\left(I, R^N\right)$ with $\beta \leq u \leq \alpha$ and $\beta \leq v \leq \alpha$, we have

$$\begin{gathered}\alpha_i'' + F_i\left(t, u, T^i u, v, T^i v\right) \geq \tfrac{1}{8}\left[(v_i - \alpha_i) + (u_i - \alpha_i)\right] \text{ on } I, \\ \alpha(0) = \alpha(2\pi),\ \alpha'(0) \geq \alpha'(2\pi).\end{gathered}$$

(B3) For all $u, v \in C\left(I, R^N\right)$ with $\beta \leq u \leq \alpha$ and $\beta \leq v \leq \alpha$, we have

$$\begin{gathered}\beta_i'' + F_i\left(t, u, T^i u, v, T^i v\right) \leq \tfrac{1}{8}\left[(v_i - \beta_i) + (u_i - \beta_i)\right] \text{ on } I, \\ \beta(0) = \beta(2\pi),\ \beta'(0) \leq \beta'(2\pi).\end{gathered}$$

(B4) $F_i(t, u, y, v, z)$ is nonincreasing in u and y and nondecreasing in v and z respectively.

(B5) $F_i\left(t, u, T^i u, u, T^i u\right) = f_i\left(t, u, T^i u\right)$, $1 \leq i \leq N$.

(B6) If there exist two functions $\rho, \gamma \in C^2\left(I, R^N\right)$ such that

$$\begin{gathered}\rho_i'' + F_i\left(t, \rho, T^i \rho, \gamma, T^i \gamma\right) = \tfrac{1}{8}(\gamma_i - \rho_i), \\ \rho(0) = \rho(2\pi),\ \rho'(0) = \rho'(2\pi),\end{gathered}$$

and

$$\begin{gathered}\gamma_i'' + F_i\left(t, \gamma, T^i \gamma, \rho, T^i \rho\right) = \tfrac{1}{8}(\rho_i - \gamma_i), \\ \gamma(0) = \gamma(2\pi),\ \gamma'(0) = \gamma'(2\pi),\end{gathered}$$

then $\rho \equiv \gamma$ on I.

Then the problem (1)-(2) has a unique solution u with $\beta(t) \leq u(t) \leq \alpha(t)$ on I.

PROOF. For a pair $(\eta, \tau) \in [\beta, \alpha] \times [\beta, \alpha]$ with $\eta \leq \tau$, consider the linear boundary value problems

$$\begin{gathered}u_i'' + \tfrac{1}{4} u_i = -F_i\left(t, \eta, T^i \eta, \tau, T^i \tau\right) + \tfrac{1}{8}(\eta_i + \tau_i), \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi),\end{gathered} \tag{3}$$

and

$$\begin{gathered}w_i'' + \tfrac{1}{4} w_i = -F_i\left(t, \tau, T^i \tau, \eta, T^i \eta\right) + \tfrac{1}{8}(\eta_i + \tau_i), \\ w(0) = w(2\pi),\ w'(0) = w'(2\pi).\end{gathered} \tag{4}$$

Note that the existence and uniqueness of the solutions to the above linear boundary value problems are guaranteed by the linear theory of ordinary differential equations. Let $V = w - u$, we have

$$\begin{aligned} V_i'' + \frac{1}{4} V_i &= F_i\left(t, \eta, T^i \eta, \tau, T^i \tau\right) - F_i\left(t, \tau, T^i \tau, \eta, T^i \eta\right), \\ V(0) &= V(2\pi),\ V'(0) = V'(2\pi). \end{aligned}$$

By the mixed monotonicity of F in (B4), the right hand side of the equation is nonnegative. Hence by Lemma 2, we get $V \geq 0$ on I. Hence we have $w \geq u$ on I. We also see that $u \geq \beta$ on I. In fact, let $W = u - \beta$ on I, we have

$$\begin{gathered}W_i'' + \tfrac{1}{4} W_i \geq 0 \\ W(0) = W(2\pi),\ W'(0) \geq W'(2\pi)\end{gathered}$$

By Lemma 2 again, we have $u \geq \beta$ on I. Similarly we have $\alpha \geq w$ on I. Thus there exists a unique solution u of (3) and a unique solution w of (4) such that $\alpha \geq w \geq u \geq \beta$ on I.

Define a map Ψ from $C\left(I, R^{2N}\right)$ into itself by $\Psi(\eta, \tau) = (u, w)$. Note that Ψ is continuous and compact on $C\left(I, R^{2N}\right)$. Let $\left(\beta^0, \alpha^0\right) = (\beta, \alpha)$ and $\left(\beta^{n+1}, \alpha^{n+1}\right) = \Psi(\beta^n, \alpha^n)$ for $n \geq 0$. We generate two sequences of functions, $\{\beta^n\}$ and $\{\alpha^n\}$ such that $\beta = \beta^0 \leq \beta^1 \leq .... \leq \beta^n \leq ... \leq \alpha^n \leq ... \leq \alpha^1 \leq \alpha^0 = \alpha$ on I. Since $\{\beta^n\}$ and $\{\alpha^n\}$ are uniformly bounded on I, there exist two convergent subsequences $\{\beta^{n_k}\}$ and $\{\alpha^{n_k}\}$ with $\{\beta^{n_k}\} \to \beta_*$ and $\{\alpha^{n_k}\} \to \alpha^*$ uniformly on $[0, 2\pi]$ as $n_k \to \infty$. By monotonicity of $\{\beta^n\}$ and $\{\alpha^n\}$, we then have $\{\beta^n\} \to \beta_*$ and $\{\alpha^n\} \to \alpha^*$ as $n \to \infty$. Hence $\Psi(\beta_*, \alpha^*) = (\beta_*, \alpha^*)$. By (B6), $\beta_* \equiv \alpha^*$ on I. (B5) implies that α^* is a solution of (1)-(2).

Remark: The arguments in [1] and Theorem 3 are similar. The only difference between them is that the maximum principle, which is used in [1], is not applicable in theorem 3. Hence Lemma 2 is used instead.

Some sufficient conditions for the existence of F in Theorem 3 are given below.

THEOREM 4. Assume that there exist two functions $\alpha, \beta \in C^2\left(I, R^N\right)$ satisfying the following conditions :

(C1) $\beta(t) \leq \alpha(t)$ on I,

(C2) For all $u \in C\left(I, R^N\right)$ with $\beta \leq u \leq \alpha$ on I, we have

$$\begin{gathered}\alpha_i'' + f_i\left(t, u, T^i u\right) \geq \tfrac{1}{8}(u_i - \alpha_i) \text{ on } I, \\ \alpha(0) = \alpha(2\pi),\ \alpha'(0) \geq \alpha'(2\pi).\end{gathered}$$

(C3) For all $u \in C\left(I, R^N\right)$ with $\beta \leq u \leq \alpha$, we have

$$\begin{gathered}\beta_i'' + f_i\left(t, u, T^i u\right) \leq \tfrac{1}{8}(u_i - \beta_i) \text{ on } I, \\ \beta(0) = \beta(2\pi),\ \beta'(0) \leq \beta'(2\pi).\end{gathered}$$

(C4) $f_i(t, u, y)$ is nonincreasing in u and y for each fixed $t \in I$.

Then the boundary value problem (1), (2) has a unique solution u with $\beta(t) \leq u(t) \leq \alpha(t)$ on I.

PROOF. Define

$$F_i(t, u, y, v, z) = \frac{1}{2}\left[f_i(t, u, y) + f_i(t, v, z)\right], t \in I, u, y, v, z \in R^N.$$

In view of the assumptions (C1)-(C4), it is not difficult to see that (B1)-(B5) are satisfied. (B6) is true as follows. If the assumption of (B6) holds, let $V = \rho - \gamma$ on I, then we have

$$\begin{gathered}V_i'' + \tfrac{1}{4}V_i = 0 \text{ on } I, \\ V(0) = V(2\pi),\ V'(0) = V'(2\pi).\end{gathered}$$

The general solution of $V_i'' + \frac{1}{4}V_i = 0$ is $V_i(t) = A_i \sin\frac{1}{2}t + B_i \cos\frac{1}{2}t$ for arbitrary constants A_i and B_i. From $V(0) = V(2\pi)$, we get $B_i = 0$ and from $V'(0) = V'(2\pi)$, we get $A_i = 0$. Hence $V = 0$ on I and then $\rho = \gamma$ on I. By Theorem 3, the problem (1)-(2) has a unique solution u with $\beta(t) \leq u(t) \leq \alpha(t)$ on I.

In a similar manner, a variant of theorem 3 can be obtained as follows.

THEOREM 5. Assume that there exist two functions $\alpha, \beta \in C^2\left(I, R^N\right)$ and a function $F \in C\left(I \times R^N \times R^N \times R^N \times R^N, R^N\right)$ such that the following conditions hold :

(D1) $\beta(t) \le \alpha(t)$ on I.

(D2)

$$\begin{gathered} \alpha_i'' + F_i\left(t, \alpha, T^i\alpha, \beta, T^i\beta\right) \ge 0 \text{ on } I, \\ \alpha(0) = \alpha(2\pi),\ \alpha'(0) \ge \alpha'(2\pi). \end{gathered}$$

(D3)

$$\begin{gathered} \beta_i'' + F_i\left(t, \beta, T^i\beta, \alpha, T^i\alpha\right) \le 0 \text{ on } I, \\ \beta(0) = \beta(2\pi),\ \beta'(0) \le \beta'(2\pi). \end{gathered}$$

(D4) For fixed t, u, y, we have

$$F_i(t, u, y, v, z) \ge F_i(t, u, y, v^*, z^*) \text{ for } v \ge v^*, z \ge z^*,$$

and for fixed t, v, z,

$$F_i(t, u^*, y^*, v, z) - F_i(t, u, y, v, z) - \frac{1}{4}(u_i^* - u_i) \le 0 \text{ for } u^* \ge u, y^* \ge y.$$

(D5) $F_i\left(t, u, T^iu, u, T^iu\right) = f_i\left(t, u, T^iu\right)$, $1 \le i \le N$.

(D6) If there exist two functions $\rho, \gamma \in C^2\left(I, R^N\right)$ such that

$$\begin{gathered} \rho_i'' + F_i\left(t, \rho, T^i\rho, \gamma, T^i\gamma\right) = 0 \text{ on } I, \\ \rho(0) = \rho(2\pi),\ \rho'(0) = \rho'(2\pi), \end{gathered}$$

and

$$\begin{gathered} \gamma_i'' + F_i\left(t, \gamma, T^i\gamma, \rho, T^i\rho\right) = 0 \text{ on } I, \\ \gamma(0) = \gamma(2\pi),\ \gamma'(0) = \gamma'(2\pi), \end{gathered}$$

then $\rho \equiv \gamma$ on I.

Then the problem (1)-(2) has a unique solution u with $\beta(t) \le u(t) \le \alpha(t)$ on I.

PROOF. Given a pair $(\eta, \tau) \in [\beta, \alpha] \times [\beta, \alpha]$ with $\eta \le \tau$, consider the linear boundary value problems

$$\begin{gathered} u_i'' + \tfrac{1}{4}u_i = -F_i\left(t, \eta, T^i\eta, \tau, T^i\tau\right) + \tfrac{1}{4}\eta_i \text{ on } I, \\ u(0) = u(2\pi),\ u'(0) = u'(2\pi), \end{gathered} \tag{5}$$

and

$$\begin{gathered} w_i'' + \tfrac{1}{4}w_i = -F_i\left(t, \tau, T^i\tau, \eta, T^i\eta\right) + \tfrac{1}{4}\tau_i \text{ on } I, \\ w(0) = w(2\pi),\ w'(0) = w'(2\pi). \end{gathered} \tag{6}$$

As in the proof of Theorem 3, by (D2)-(D4) and Lemma 2, there exists a unique solution u of (5) and solution w of (6) such that $\alpha \ge w \ge u \ge \beta$ on I. Define a map Ψ from $C\left(I, R^{2N}\right)$ into itself by $\Psi(\eta, \tau) = (u, w)$. Then Ψ is continuous and compact on $C\left(I, R^{2N}\right)$. Let $\left(\beta^0, \alpha^0\right) = (\beta, \alpha)$ and $\left(\beta^{n+1}, \alpha^{n+1}\right) = \Psi(\beta^n, \alpha^n)$ for $n \ge 0$. Two sequences of functions $\{\beta^n\}$ and $\{\alpha^n\}$ are generated such that

$$\beta = \beta^0 \le \beta^1 \le .... \le \beta^n \le ... \le \alpha^n \le ... \le \alpha^1 \le \alpha^0 = \alpha \text{ on } I.$$

By uniform boundedness and monotonicity of $\{\beta^n\}$ and $\{\alpha^n\}$, we see that $\{\beta^n\} \to \beta_*$ and $\{\alpha^n\} \to \alpha^*$ as $n \to \infty$ for some β_* and α^* in $C\left(I, R^N\right)$ and $\Psi\left(\beta_*, \alpha^*\right) = \left(\beta_*, \alpha^*\right)$. By (D6) , $\beta_* \equiv \alpha^*$ on I. Hence by (D5), we get the solution of (1)-(2).

THEOREM 6. Assume that there exist two functions $\alpha, \beta \in C^2\left(I, R^N\right)$ satisfying the following conditions :

(E1) $\beta(t) \leq \alpha(t)$ on I.

(E2)

$$\begin{gathered} \alpha_i'' + f_i\left(t, \alpha, T^i\alpha\right) \geq 0 \text{ on } I, \\ \alpha(0) = \alpha(2\pi),\ \alpha'(0) \geq \alpha'(2\pi). \end{gathered}$$

(E3)

$$\begin{gathered} \beta_i'' + f_i\left(t, \beta, T^i\beta\right) \leq 0 \text{ on } I, \\ \beta(0) = \beta(2\pi),\ \beta'(0) \leq \beta'(2\pi). \end{gathered}$$

(E4) For $u^* \geq u, y^* \geq y$, we have

$$f_i(t, u^*, y^*) - f_i(t, u, y) - \frac{1}{4}\left(u_i^* - u_i\right) \leq 0.$$

Then the boundary value problem (1)-(2) has a unique solution u with $\beta(t) \leq u(t) \leq \alpha(t)$ on I.

PROOF. Let

$$F_i(t, u, y, v, z) = \frac{1}{2}\left[f_i(t, u, y) + f_i(t, v, z) + \frac{1}{4}\left(u_i - v_i\right)\right] \text{ on } I.$$

All hypotheses of theorem 5 are satisfied. Hence there exists a unique solution u of the problem (1)-(2) on I.

References

[1] S. W. Yu, On nonlinear integro-differential equations, Master Thesis, Chengchi Univ., Taiwan, 1989.

[2] C. de Coster and P. Habets, Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results, Nonlinear analysis and boundary value problems for ordinary differential equations. (edited by F. Zanolin), C.I.S.M. Courses and Lectures, Vol. 371, Springer -Verlag, 1993, 1–79.

[3] M. Khavanin, The method of mixed momotony and second order integro-differential system, Appl. Anal., 28(1998), 199–206.

[4] R. Kannan and V. Lakshmikantham, Existence of periodic solutions of nonlinear boundary value problems and the method of upper and lower solutions., Appl. Anal., 17(1984), 103–113.