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Existence of Absorbing Set for a Nonlinear Wave Equation *

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Abstract

We prove the existence of an absorbing set for a Cauchy problem involving a nonlinear wave equation.

Let $u_0 \in W_2^{(1)}(0,l), u_1$and h are given elements in $L_2(0,l)$, α, γ and c be positive numbers and $f(.) \in C^1(R)$ such that

$$\mathcal{F}(s) = \int_0^s f(\eta)d\eta \geq -c, \tag{1}$$

and

$$f(s)s - \mathcal{F}(s) \geq -c \tag{2}$$

for all $s \in R$, where $W_2^{(1)}(0,l) = \{u : u, u' \in L_2(0,l), u(0) = u(l) = 0\}$. We consider the following Cauchy problem:

$$u_{tt} - u_{xx} + \alpha u_{tx} + \gamma u_t + f(u) = h(x),\ x \in (0,l)\,,\ t \in R^+ \tag{3}$$

$$u\,(x,0) = u_0(x),\ u_t\,(x,0) = u_1\,(x)\,,\ x \in (0,l) \tag{4}$$

$$u\,(0,t) = u\,(l,t) = 0,\ t \in R^+. \tag{5}$$

A continuous semigroup of operators $S\,(t)$ from $X = W_2^{(1)}(0,l) \times L_2\,(0,l)$ into itself can be defined and satisfies the semigroup condition $S\,(t+s) = S\,(t)\,S\,(s)$. We will prove the existence of an absorbing set in X by defining a Lyapunov like functional $\phi\,(u, u_t)$. The methods used are inspired from the results of Ladyzhenskaya [3, 4] and used in Eden and Kalantarov [1, 2]. In the following we shall use the following notations:

$$\|u\|^2 = \int_0^l u^2(x)dx,$$

and

$$(u,v) = \int_0^l u(x)v(x)dx.$$

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We also use the Wirtinger inequality

$$\int_0^l u^2(x)dx \leq \lambda \int_0^l u_x^2(x)dx \tag{6}$$

where $\lambda = l^2/\pi^2$.

THEOREM 1. Let $u_0, u_1, h, \alpha, \gamma, c$ be given. Suppose $f \in C^1(R)$ satisfies the conditions (1) and (2). Suppose further that the problem (3)-(5) has a unique weak solution. Then the semigroup $S(t)$, $t > 0$, defined by $S(t)(u_0, u_1) = (u(t), u_1(t))$ generated by the problem (3)-(5) is bounded and dissipative.

PROOF. Suppose $u(x,t)$ is a weak solution of (3)-(5). Let δ be a positive number. Multiplying the equation (3) by $u_t + \delta u$ and integrating over $(0,l)$ we get

$$\begin{aligned} 0 \;=\; & \frac{d}{dt}(\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u_x\|^2 + (\mathcal{F}(u),1) - (h,u) + \delta\,(u,u_t) \\ & + \frac{\delta\gamma}{2}\|u\|^2) + \gamma\|u_t\|^2 - \alpha\delta\,(u_x,u_t) + \delta\|u_x\|^2 \\ & - \delta\|u_t\|^2 + \delta(f(u),u) - \delta(h,u). \end{aligned} \tag{7}$$

We consider the functional

$$\phi\,(u,u_t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u_x\|^2 + (\mathcal{F}(u),1) - (h,u) + \delta\,(u,u_t) + \frac{\delta\gamma}{2}\|u\|^2. \tag{8}$$

From (7) we write

$$\frac{d}{dt}\phi\,(u,u_t) = -\gamma\|u_t\|^2 + \alpha\delta\,(u_x,u_t) - \delta\|u_x\|^2 + \delta\|u_t\|^2 - \delta(f(u),u) + \delta(h,u). \tag{9}$$

Let η be a positive number such that $\eta < \delta$. Then from (7) we write

$$\begin{aligned} & \frac{d}{dt}\phi\,(u,u_t) + \eta\phi\,(u,u_t) \\ = \; & \frac{\eta}{2}\|u_t\|^2 + \frac{\eta}{2}\|u_x\|^2 + \eta(\mathcal{F}(u),1) - \eta(h,u) + \delta\eta\,(u,u_t) + \frac{\delta\eta\gamma}{2}\|u\|^2 \\ & -\gamma\|u_t\|^2 + \alpha\delta\,(u_x,u_t) - \delta\|u_x\|^2 + \delta\|u_t\|^2 - \delta(f(u),u) + \delta(h,u). \end{aligned} \tag{10}$$

By using (6) we obtain

$$\delta\eta\,|(u,u_t)| \leq \frac{\delta\eta}{2}\|u_t\|^2 + \frac{\delta\eta l^2}{2\pi^2}\|u_x\|^2, \tag{11}$$

$$(\delta-\eta)\,|(h,u)| \leq \frac{\eta}{2}\|u_x\|^2 + \frac{l^2(\delta-\eta)^2}{2\eta\pi^2}\|h\|^2, \tag{12}$$

and

$$\alpha\delta\,|(u_x,u_t)| \leq \frac{\gamma}{2}\|u_t\|^2 + \frac{\delta^2\alpha^2}{2\gamma}\|u_x\|^2. \tag{13}$$

With the help of the inequalities (11), (12) and (13) we obtain from (10) that

$$\begin{aligned}
&\frac{d}{dt}\phi\left(u,u_t\right)+\eta\phi\left(u,u_t\right)\\
\leq\quad&\left(\delta-\frac{\gamma}{2}+\frac{\delta\eta}{2}+\frac{\eta}{2}\right)\left\|u_t\right\|^2+\left(\eta+\frac{\delta\eta l^2}{2\pi^2}+\frac{\delta\eta\gamma l^2}{2\pi^2}+\frac{\delta^2\alpha^2}{2\gamma}-\delta\right)\left\|u_x\right\|^2\\
&-\delta((f(u),u)-(\mathcal{F}(u),1))-(\delta-\eta)\left(\mathcal{F}(u),1\right)+\frac{l^2(\delta-\eta)^2}{2\eta\pi^2}\left\|h\right\|^2. \qquad (14)
\end{aligned}$$

Thus by choosing

$$\delta<\min\left\{\frac{\gamma}{4},\frac{\gamma}{\alpha^2}\right\},$$

$$\eta<\min\left\{\frac{2\gamma}{\gamma+4},\frac{\gamma\pi^2}{2\pi^2\alpha^2+\gamma^2l^2+\gamma l^2}\right\},$$

and writing

$$(\delta-\eta)\left(\mathcal{F}(u),1\right)\geq -c\left(\delta-\eta\right)l,$$

$$\delta((f(u),u)-(\mathcal{F}(u),1))\geq -c\delta l,$$

we get from (14) that

$$\frac{d}{dt}\phi\left(u,u_t\right)+\eta\phi\left(u,u_t\right)\leq c_1 \qquad (15)$$

where

$$c_1=\frac{l^2(\delta-\eta)^2}{2\eta\pi^2}\left\|h\right\|^2+cl\left(2\delta-\eta\right).$$

It follows from (15) that

$$\frac{d}{dt}\left(e^{\eta t}\phi\left(u,u_t\right)\right)\leq e^{\eta t}c_1 \qquad (16)$$

and

$$\phi\left(u\left(.,t\right),u_t\left(.,t\right)\right)\leq\phi\left(u\left(\cdot,0\right),u_t\left(\cdot,0\right)\right)e^{-\eta t}+\frac{c_1}{\eta}. \qquad (17)$$

By using (8) and (1) and writing inequalities similar to (11) and (12), we may obtain

$$\begin{aligned}
\phi\left(u\left(.,t\right),u_t\left(.,t\right)\right)\quad\geq\quad&\left(\frac{1}{2}-\frac{\delta\gamma l^2}{2\pi^2}-\frac{\delta l^2}{2\pi^2}-\frac{\delta}{2}\right)\left\|u_x\right\|^2\\
&+\left(\frac{1}{2}-\frac{\delta}{2}\right)\left\|u_t\right\|^2-\left(cl+\frac{l^2}{2\delta\pi^2}\left\|h\right\|^2\right). \qquad (18)
\end{aligned}$$

If we choose

$$\delta<\min\left\{1,\frac{\pi^2}{l^2\gamma+l^2+\pi^2}\right\},$$

then we get from (18) that

$$\phi\left(u\left(.,t\right),u_t\left(.,t\right)\right)\geq a_0\left(\left\|u_x\right\|^2+\left\|u_t\right\|^2\right)-c_2$$

where

$$a_0 = \frac{1}{2}\min\left\{1 - \frac{\delta l^2}{\pi^2}(\gamma+1) - \delta, 1-\delta\right\}$$

and

$$c_2 = \frac{l^2}{2\delta\pi^2}\|h\|^2 + cl.$$

If we use this result in (17) we obtain

$$\|u_x\|^2 + \|u_t\|^2 \leq \frac{e^{-\eta t}}{a_0}\phi\left(u(\cdot,0), u_t(\cdot,0)\right) + \frac{1}{a_0}\left(\frac{c_1}{\eta} + c_2\right).$$

Then for $R = \frac{1}{a_0}\left(\frac{c_1}{\eta} + c_2\right)$, $B = \{\{u,v\} \in X : \|\{u,v\}\|_X \leq \sqrt{2R}\}$ is the absorbing set for the semigroup $S(t)$ in X. Thus the semigroup $S(t)$, $t > 0$, is bounded and dissipative.

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