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Existence of Positive Solutions for Singular Second Order Boundary Value Problems *†

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Abstract

In this paper, using fixed point theorem in cones and a transformation $y(t) = \int_0^t \frac{1}{p(s)} z(s)\, ds$, we establish some existence results for singular second order boundary value problems of the form

$$\left(py'\right)' + p(t)\, q(t)\, f\left(t, y, py'\right) = 0,\ 0 < t < 1,$$

where $f(t, y, z)$ is allowed to be singular at $y = 0$ and $z = 0$.

1 Introduction

This paper is devoted to the study of the existence of positive solutions for singular second order boundary value problems of the form

$$\begin{cases} (py')' + p(t)\, q(t)\, f(t, y, py') = 0, & 0 < t < 1, \\ y(0) = \lim_{t \to 1^-} p(t)\, y'(t) = 0, \end{cases} \tag{1}$$

where $\lim_{y \to 0^+} f(t, y, z) = +\infty$ and $\lim_{z \to 0^+} f(t, y, z) = +\infty$ uniformly on compact subset of $[0, 1] \times (0, +\infty)$. That is, we will allow our nonlinear term f to be singular at $y = 0$ and $z = 0$.

In [1], Erbe and Wang study the existence of positive solutions of the equation $u'' + a(t)\, f(u) = 0$ by using the Krasnosel'skii fixed point theorem [2], where $a(t)$ is continuous on $[0, 1]$ and $f(u)$ is continuous on $[0, \infty)$. Krasnosel'skii fixed point theorem has been widely used to discuss the existence of positive solutions for boundary value problems. In [3-5], O'Regan et al. showed the existence of positive solutions for singular second order differential equations of the form

$$(py')' + p(t)\, q(t)\, f(t, y, py') = 0,\ 0 < t < 1,$$

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where $f(t, y, z)$ is continuous on $[0, 1] \times R^2$. In [6], its authors used the nonlinear alternative of Leray and Schauder to prove the existence results for singular second order boundary value problems of the form

$$(py')' + p(t)\,q(t)\,f(t, y) = 0,$$

here $\lim_{y\to 0^+} f(t, y) = +\infty$. In [7], by using an upper and lower solution approach, O'Regan and Agarwal presented the existence results for singular problems of the from

$$\begin{cases} (py')' + p(t)\,q(t)\,f(t, y, py') = 0, & 0 < t < 1, \\ y(1) = \lim_{t\to 0^+} p(t)\,y'(t) = 0, \end{cases} \tag{2}$$

where f is allowed to be singular at $y = 0$. When f is singular at $y = 0$ and $z = 0$, few people (e.g. [9]) studied the problem (2). In this paper, with the use of certain fixed point theorem in cones and a transformation

$$y(t) = \int_0^t \frac{1}{p(s)} z(s)ds,$$

we will show the existence of positive solutions for the problem (1). Our results are different from that in [9] and simpler than that in [7].

2 Main Results

Let $py' = z(t)$. We can transform (1) into

$$\begin{cases} py' = z(t), \\ z'(t) + p(t)q(t)f(t, y, z) = 0, & 0 < t < 1, \\ y(0) = \lim_{t\to 1^-} z(t) = 0. \end{cases} \tag{3}$$

Consequently (3) is equivalent to the fixed point problem

$$(Tz)(t) = \int_t^1 p(s)q(s)f(s, (Az)(s), z(s))\,ds, \tag{4}$$

where

$$y(t) = \int_0^t \frac{1}{p(s)} z(s)ds = (Az)(t). \tag{5}$$

We will suppose that the following conditions are satisfied:

(H_1) $f : [0, 1] \times (0, +\infty) \times (0, +\infty) \to (0, +\infty)$ is continuous, $\lim_{y\to 0^+} f(t, y, z) = +\infty$ and $\lim_{z\to 0^+} f(t, y, z) = +\infty$ uniformly on bounded subsets of $[0, 1] \times (0, +\infty)$;

(H_2) $p(t) \in C[0, 1] \cap C^1(0, 1)$ with $p > 0$ on $(0, 1)$;

(H_3) $q(t) \in C(0, 1)$ with $q > 0$ on $(0, 1)$;

(H_4) $\int_0^1 \frac{1}{p(s)} ds < +\infty$, $\int_0^1 p(s)q(s)ds < +\infty$, and $\lim_{t\to 1^-} p(t)q(t)f(t, y, z) = +\infty$ uniformly on bounded subsets of $(0, +\infty) \times (0, +\infty)$;

(H_5) $f(t, y, z) \le h(y)g(z)$ for $(t, y, z) \in [0, 1] \times (0, +\infty) \times (0, +\infty)$, where $g, h \in C((0, +\infty), (0, +\infty))$;

(H_6) p^2q is bounded on $[0,1]$ and there exists $R>0$ such that

$$\int_0^{(R+1)\int_0^1 \frac{1}{p(s)}ds+1} h(u)du < +\infty$$

and

$$\int_0^R \frac{u}{g(u)}du > \sup_{t\in[0,1]} p^2(t)q(t) \int_0^{(R+1)\int_0^1 \frac{1}{p(s)}ds+1} h(u)du;$$

(H_7) $\int_0^1 p(t)q(t)\max g[1-t,R]dt < +\infty$ and

$$\int_0^1 p(t)q(t)\max h\left[\int_0^t \frac{1-t_0}{P(s)}ds, R\int_0^1 \frac{1}{p(s)}ds+1\right]dt < +\infty$$

for each $t_0 \in [0,1)$, where $\max g[a,b] = \max_{a\le x\le b} g(x), a \le b$.

We will need the following lemma, its proof can be seen in [8].

LEMMA 1. Let K be a cone of the Banach space E, $B_R(0) = \{x \in K : ||x|| \le R\}$, and $F : B_R(0) \to K$ is a completely continuous operator. In addition suppose

(i) $F(x) \neq \lambda x$ for $||x|| = R$, $\lambda > 1$,

(ii) there exists $r \in (0,R)$ such that $F(x) \neq \lambda x$ for $||x|| = r, 0 < \lambda < 1$,

(iii) $\inf\{||Fx|| : ||x|| = r\} > 0$.

Then F has at least one fixed point on $r \le ||x|| \le R$.

Consider the problem

$$\begin{cases} (py')' + p(t)\,q(t)\,f(t,y,py') = 0, & 0<t<1, \\ y(0) = \lim_{t\to 1^-} p(t)\,y'(t) = 1/m. \end{cases} \tag{6}$$

where $m \in N$, which is equivalent to the fixed point problem

$$T_m z(t) = \int_t^1 f\left(s, (Az)(s) + \frac{1}{m}, z(s)\right) p(s)q(s)ds + \frac{1}{m}. \tag{7}$$

Let

$$D[0,1] = \{z \in C([0,1],[0,+\infty)) : z \text{ is nonincreasing on } [0,1]\},$$

then $D[0,1]$ is a cone of Banach space $C[0,1]$. For $z(t) \in D[0,1]$, we define

$$Iz(t) = \begin{cases} z(t), & z(1) \ge 1/m, \\ z(t) + \left(\frac{1}{m} - z(1)\right), & z(1) < 1/m. \end{cases} \tag{8}$$

LEMMA 2. Suppose (H_1)-(H_4) hold, then $T_m I$ is a completely continuous operator on $D[0,1]$.

PROOF. First we show $T_m I$ is a continuous operator on $D[0,1]$. Let $z, z_n \in D[0,1]$ such that $z_n \to z$. Since f is uniformly continuous on compact subsets of $[0,1] \times [1/m,+\infty) \times [1/m,+\infty)$, then for each $\varepsilon > 0$, there is N such that

$$\left| f\left(s, (AIz)(s) + \frac{1}{m}, Iz(s)\right) - f\left(s, (AIz_n)(s) + \frac{1}{m}, Iz_n(s)\right)\right| < \varepsilon$$

when $n > N$, $s \in [0,1]$. This together with (H_4) gives

$$\begin{aligned}
& |T_m Iz(t) - T_m Iz_n(t)| \\
\leq & \int_t^1 \left| f\left(s, (AIz)(s) + \frac{1}{m}, Iz(s)\right) - f\left(s, (AIz_n)(s) + \frac{1}{m}, Iz_n(s)\right)\right| p(s)q(s)ds \\
\leq & \int_0^1 \left| f\left(s, (AIz)(s) + \frac{1}{m}, Iz(s)\right) - f\left(s, (AIz_n)(s) + \frac{1}{m}, Iz_n(s)\right)\right| p(s)q(s)ds \\
\leq & \ \varepsilon \int_0^1 p(s)q(s)ds,
\end{aligned}$$

for $n > N$, $t \in [0,1]$. We obtain that $T_m I$ is a continuous operator on $D[0,1]$.

Next we show $T_m I$ is a compact map. Let $\Omega \subseteq D[0,1]$ be bounded, that is that there exists a constant M with $||z|| \leq M$ for each $z \in \Omega$. By using (H_1) and (H_4), there is $M' > 0$ such that $|f(s, (AIz)(t) + 1/m, Iz(t))| \leq M'$ for each $z \in \Omega, t \in [0,1]$. Therefore, $|T_m Iz| \leq M' \int_0^1 p(s)q(s)ds$ for each $z \in \Omega$, that is $T_m I\Omega$ is completely bounded.

For each $z \in \Omega, t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we have

$$\begin{aligned}
|T_m Iz(t_1) - T_m Iz(t_2)| &= \int_{t_1}^{t_2} f\left(s, (AIz)(s) + \frac{1}{m}, Iz(s)\right) p(s)q(s)ds \\
&\leq M' \int_{t_1}^{t_2} p(s)q(s)ds.
\end{aligned}$$

(H_4) and the above inequality imply that $T_m I\Omega$ is equicontinuous. Consequently the Arzela-Ascoli theorem implies $T_m I\Omega$ is relatively compact. So $T_m I$ is a completely continuous. The proof is complete.

THEOREM 1. Suppose (H_1)-(H_7) hold, then (1) has a positive solution $y \in C^1[0,1] \cap C^2(0,1)$ with $py' \in C[0,1]$.

PROOF. Take R as in (H_6). First we show that $T_m Iz \neq \mu z$ for each $||z|| = R, \mu > 1$. If this is not true, then there exist $\lambda \in (0,1)$ and $z \in D[0,1]$ with $||z|| = R$ such that $\lambda T_m Iz = z$, that is

$$\lambda \int_t^1 f\left(s, (AIz)(s) + \frac{1}{m}, Iz(s)\right) p(s)q(s)ds + \frac{\lambda}{m} = z.$$

So $z(0) = ||z|| = R$, $z(1) = \lambda/m$,

$$\begin{aligned}
-z'(t) &= \lambda f\left(t, (AIz)(t) + \frac{1}{m}, Iz(t)\right) p(t)q(t) \\
&\leq \lambda h\left((AIz)(t) + \frac{1}{m}\right) g\left(Iz(t)\right) p(t)q(t).
\end{aligned}$$

Let $y(t) = (AIz)(s) + 1/m$, we have

$$-(py')'py' \leq \lambda h(y(t))g(py')p(t)q(t)py',\ t \in (0,1),$$

and integration from 0 to 1 yields

$$\begin{aligned}\int_{\frac{\lambda}{m}}^{R} \frac{u}{g(u)}du &\leq \lambda \sup_{t\in[0,1]} p^2(t)q(t)\int_0^1 h(y(t))y'(t)dt \\ &\leq \lambda \sup_{t\in[0,1]} p^2(t)q(t)\int_{\frac{1}{m}}^{(R+1)\int_0^1 \frac{ds}{p(s)}+\frac{1}{m}} h(u)du \\ &\leq \lambda \sup_{t\in[0,1]} p^2(t)q(t)\int_0^{(R+1)\int_0^1 \frac{ds}{p(s)}+\frac{1}{m}} h(u)du\end{aligned}$$

If m is sufficiently large, (H_6) implies

$$\int_{\frac{\lambda}{m}}^{R} \frac{u}{g(u)}du > \sup_{t\in[0,1]} p^2(t)q(t)\int_0^{(R+1)\int_0^1 \frac{ds}{p(s)}+\frac{1}{m}} h(u)du.$$

This is a contradiction. Thus $T_mIz \neq \mu z$ for each $||z|| = R, \mu > 1$.

Next we show that there is $r \in (0, R)$ such that $T_mIz \neq \lambda z$ for each $||z|| = r, \lambda \in (0, 1)$. Since $\lim_{z\to 0^+} f(t, y, z) = +\infty$ uniformly on bounded subsets of $[0, 1] \times (0, +\infty)$, then there is a sufficiently small $r > 0$ such that

$$||T_mIz|| = \int_0^1 f\left(s, (AIz)(s), Iz(s)\right) p(s)q(s)ds + \frac{1}{m} > r.$$

If $||z|| = r$ and $m \to +\infty$, Then $T_mIz \neq \lambda z$ for $||z|| = r$ and $\lambda \in (0, 1)$. By Lemma 1 and 2, there is $M > 0$ such that T_mI has a fixed point z_m on $D[0, 1]$ with $r \leq ||z_m|| \leq R$ when $m > M$, and $z_m(t) \geq 1/m$ for $t \in [0, 1]$. Therefore, z_m is a fixed point of T_m.

It is clear that $\{z_m\}$ is completely bounded. Next we show $\{z_m\}$ is equicontinuous. (H_4) implies that there is $t_0 \in [0, 1)$ such that $p(t)q(t)f(t, y, z) \geq 1$ on $[t_0, 1] \times (0, R\int_0^1 \frac{1}{p(s)}ds + 1] \times (0, R]$. Thus

$$z_m(t) = \int_t^1 f\left(s, (Az_m)(s) + \frac{1}{m}, z_m(s)\right) p(s)q(s)ds + \frac{1}{m} > 1 - t,\ t \in [t_0, 1], \quad (9)$$

$$z_m(t) = \int_t^1 f\left(s, (Az_m)(s) + \frac{1}{m}, z_m(s)\right) p(s)q(s)ds + \frac{1}{m} > 1 - t_0\ t \in [0, t_0], \quad (10)$$

$$(Az_m)(t) + \frac{1}{m} = \int_0^t \frac{1}{p(s)}z_m(s)ds + \frac{1}{m} > \int_0^t \frac{1 - t_0}{p(s)}ds,\ t \in [0, t_0], \quad (11)$$

and

$$(Az_m)(t) + \frac{1}{m} = \int_0^t \frac{1}{p(s)}z_m(s)ds + \frac{1}{m} > \int_0^{t_0} \frac{1 - t_0}{p(s)}ds,\ t \in [t_0, 1]. \quad (12)$$

Since

$$\begin{aligned}0 &\leq -z_m'(t) = p(t)q(t)f(t, (Az_m)(t) + \frac{1}{m}, z_m(t)) \\ &\leq p(t)q(t)h((Az_m)(t) + \frac{1}{m})g(z_m(t)),\end{aligned}$$

so we have

$$0 \le -z_m'(t) = p(t)q(t)\max h\left[\int_0^t \frac{1-t_0}{p(s)}ds, R\int_0^1 \frac{1}{p(s)}ds+1\right]\max g[1-t_0,R]$$

for $t \in [0,t_0]$,

$$0 \le -z_m'(t) = p(t)q(t)\max h\left[\int_0^{t_0} \frac{1-t_0}{p(s)}ds, R\int_0^1 \frac{1}{p(s)}ds+1\right]\max g[1-t,R]$$

for $t \in [t_0,1]$. Thus the equicontinuity of $\{z_m\}$ follows from (H_7) and the above inequalities. Consequently the Arzela-Ascoli theorem guarantees the existence of a subset N_0 of $\{M+1, M+2, ...\}$ and a function $z \in D[0,1]$ with z_m converging uniformly on $[0,1]$ to z as $m \to +\infty$ through N_0. Also $z(0)=0$, (9) and (10) imply $z(t)>0$ for $t \in [0,1)$. Thus $(Az_m)(t) \to \int_0^t (z(s)/p(s))ds$ uniformly on $[0,1]$ as $m \to +\infty$ through N_0. Now $z_m, m \in N_0$, satisfies the integral equation

$$z_m(t) = \int_t^1 f\left(s, (Az_m)(s) + \frac{1}{m}, z_m(s)\right)p(s)q(s)ds.$$

Fix $t \in (0,1)$, we have $f\left(s,(Az_m)(s)+\frac{1}{m}, z_m(s)\right) \to f\left(s,(Az)(s),z(s)\right)$ uniformly on compact subsets of $[t,1)$, so letting $m \to \infty$ through N_0 gives

$$z(t) = \int_t^1 f\left(s,(Az)(s),z(s)\right)p(s)q(s)ds.$$

Let $y(t) = \int_0^t \frac{z(s)}{p(s)}ds$, then $y(t)$ is a solution of (1) with $y \in C^1[0,1] \cap C^2(0,1)$, and $py' \in C[0,1]$. The proof is complete.

REMARK**:** Notice (H_6) can be replaced by

$$\exists r, 1<r<+\infty, \int_0^{(R+1)\int_0^1 \frac{1}{p(s)}ds+1} h^r(u)du < +\infty,\ \int_0^1 \left[p^{\frac{r+1}{r}}(u)q(u)\right]^{\frac{r}{r-1}}du < +\infty,$$

and

$$\int_0^R \frac{u^{\frac{1}{r}}}{g(u)}du > \left(\int_0^{(R+1)\int_0^1 \frac{1}{p(s)}ds+1} h^r(u)du\right)^{\frac{1}{r}}\left(\int_0^1 \left[p^{\frac{r+1}{r}}(u)q(u)\right]^{\frac{r}{r-1}}du\right)^{\frac{r-1}{r}},$$

then the result in Theorem 1 is again true. To see this, notice in this case we choose $\delta > 0$ so that

$$\int_\delta^R \frac{u^{\frac{1}{r}}}{g(u)}du > \left(\int_0^{(R+1)\int_0^1 \frac{1}{p(s)}ds+1} h^r(u)du\right)^{\frac{1}{r}}\left(\int_0^1 \left[p^{\frac{r+1}{r}}(u)q(u)\right]^{\frac{r}{r-1}}du\right)^{\frac{r-1}{r}}$$

hold . Essentially the same reasoning as in the proof of Theorem 1 establishes the proof.

EXAMPLE. Consider the boundary value problem

$$\begin{cases} \left(t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}y'\right)' + t^{-\frac{1}{4}}(1-t)^{-\frac{1}{4}}y^{-\frac{1}{4}}(y')^{-\frac{1}{2}} = 0, \\ y(0) = \lim_{t\to 1^-} t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}y'(t) = 0. \end{cases} \tag{13}$$

Let $p(t) = t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}$, $q(t) = t^{-\frac{3}{4}}(1-t)^{-\frac{3}{4}}$, $f(t,y,z) = y^{-\frac{1}{4}}z^{-\frac{1}{2}}$, $h(y) = y^{-\frac{1}{4}}$, $g(z) = z^{-\frac{1}{2}}$. Clearly, all assumptions of Theorem 1 are fulfilled . Hence the problem (13) has at least one positive solution $y \in C^1[0,1] \cap C^2(0,1)$ with $py' \in C[0,1]$.

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