Applied Mathematics E-Notes, 2(2002), 45-50 © ISSN 1607-2510
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New Exact Traveling Wave Solutions for Compound KdV-Burgers Equations in Mathematical Physics *†

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Received 1 December 2001

Abstract

With the aid of Mathematica, new explicit and exact travelling wave and solitary solutions for compound KdV-Burgers equations are obtained by using an improved sine-cosine method and the Wu elimination method.

1 Introduction

In the present paper we consider the compound KdV-Burgers equation

$$u_t + puu_x + qu^2u_x + ru_{xx} - su_{xxx} = 0, \tag{1}$$

where p, q, r, s are constants. This equation can be thought of as a generalization of the KdV, mKdV and Burgers equations, involving nonlinear dispersion and dissipation effects. As particular cases, (i) when $r = 0$ and $pqs \neq 0$, (1) becomes the compound KdV equation

$$u_t + puu_x + qu^2u_x - su_{xxx} = 0, \tag{2}$$

(ii) when $p = 0$ and $qrs \neq 0$, (1) becomes the KdV-Burgers equation

$$u_t + qu^2u_x + ru_{xx} - su_{xxx} = 0, \tag{3}$$

and (iii), when $r = 0$ in (3), then we get the mKdV equation

$$u_t + qu^2u_x - su_{xxx} = 0. \tag{4}$$

In a recent paper, Wang [1] has found some exact solutions of (1) by using the homogenous balance method. In this paper we obtain new travelling wave solutions of (1) by using an improved sine-cosine method [2,3] and Wu's elimination method [4]. The main idea of the algorithm is as follows. Given a partial differential equation of the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, ...) = 0, \tag{5}$$

*Mathematics Subject Classifications: 35Q51, 35Q53, 35Q99.

†The work is supported by the National key Basic Research Development of China (Grant No.199803060) and the National NSF of China (Grant No. 10072013, 10072189).

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where P is a polynomial. By assuming travelling wave solutions of the fom

$$u(x,t) = \varphi(\xi), \xi = \lambda(x - kt + c_0), \tag{6}$$

where k, λ are constants to be determined, and c_0 is an arbitrary constant, we are led to the ordinary differential equation

$$P(\varphi, \varphi', \varphi'', ...) = 0, \tag{7}$$

where φ' denotes $d\varphi/d\xi$. According to the sine-cosine method (see [1-6] for details), we suppose that (7) has the following formal travelling wave solution

$$\varphi(\xi) = \sum_{i=1}^{n} \sin^{i-1}\omega(B_i \sin\omega + A_i \cos\omega) + A_0, \tag{8}$$

and

$$\frac{d\omega}{d\xi} = \sin\omega, \text{ or } \frac{d\omega}{d\xi} = \cos\omega, \tag{9}$$

where $A_0, ..., A_n$ and $B_1, ..., B_n$ are constants to be determined. Then we proceed as follows:

Step 1. Equating the highest order nonlinear term and highest order linear partial derivative in (5), yield the value of n.

Step 2. Substituting (8), (9) into (7), we obatin a polynomial equation involving $\cos\omega \sin^i\omega$, $\sin^i\omega$ for $i = 0, 1, 2, ...n$. This step can be carried out the help of Mathematica.

Step 3. Setting the constant term and coefficients of $\sin\omega$, $\cos\omega$, $\sin\omega\cos\omega$, $\sin^2\omega$, $\cdots$, in the equation obtained in step 2 to zero, we obtain a system of algebraic equations about the unknown numbers $k, \lambda, B_0, A_i, B_i$ for $i = 1, 2, ..., n$.

Step 4. Using Wu's elimination methods, the algebraic equations in step 3 are solved with the aid of a computer.

These then yield the solitary wave solutions for the system (5).

We remark that the above method yield solutions that include terms sechξ or tanhξ, as well as their combinations. They are different from those that are obtained by other methods, such as the homogenous balance method [5,6].

2 New Explicit Solutions

We assume formal solutions of the form

$$u(x,t) = \varphi(\xi), \xi = \lambda(x - kt + c), \tag{10}$$

where λ, k are constants to be determined later and c_0 is an arbitrary constants. Substituting (10) into (1), we obtain an ordinary differential

$$k\varphi' - p\varphi\varphi' - q\varphi^2\varphi' - \lambda r\varphi'' + s\lambda^2\varphi''' = 0. \tag{11}$$

According to the algorithm described in the previous section, we suppose that (11) has the following formal solutions

$$\varphi(\xi) = A_0 + A_1 \sin\omega + A_2 \cos\omega, \tag{12}$$

and target equation

$$\frac{d\omega}{d\xi} = \sin\omega. \tag{13}$$

With the aid of Mathematica or Maple, from (12) and (13), we can get

$$\begin{aligned} & k\varphi' - p\varphi\varphi' - q\varphi^2\varphi' - \lambda r\varphi'' + s\lambda^2\varphi''' \\ = \ & [6A_2 s\lambda^2 - q(A_2^3 - 3A_1^2 A_2)]\sin^4\omega - (6A_1 s\lambda^2 + qA_1^3 - qA_1A_2^2)\sin^3\omega\cos\omega \\ & +(2pA_1A_2 + 4qA_0A_1A_2 + 2\lambda rA_1)\sin^3\omega \\ & +[2r\lambda A_2 - P(A_1^2 - A_2^2) - q(2A_0A_1^2 - 2A_0A_2^2)]\sin^2\omega\cos\omega \\ & +[pA_0A_1 - 4A_2 s\lambda^2 - kA_2 + q(A_2A_0^2 + A_2^3 - 2A_1^2A_2)]\sin^2\omega \\ & +(s\lambda^2 A_1 + kA_1 - pA_0A_1 - qA_1A_0^2)\cos\omega\sin\omega \\ & +[-pA_1A_2 - r\lambda A_1 - 2qA_0A_1A_2]\sin\omega \\ = \ & 0. \end{aligned}$$

Setting the coefficients of $\sin^j\omega\cos^i\omega$ for $i = 0, 1$ and $j = 1, 2, 3, 4$ to zero, we have the following set of overdetermined equations in the unknowns $A_0, A_1, A_2, \lambda, k$:

$$\begin{gathered} 6A_2 s\lambda^2 - q(A_2^3 - 3A_1^2A_2) = 0, \\ 6A_1 s\lambda^2 + q(A_1^3 - 3A_1^2A_2) = 0, \\ 2pA_1A_2 + 4qA_0A_1A_2 + 2\lambda rA_1 = 0 \\ 2r\lambda A_2 - p(A_1^2 - A_2^2) - q(2A_0A_1^2 - 2A_0A_2^2) = 0 \\ pA_0A_1 - 4A_2 s\lambda^2 - kA_2 + q(A_2A_0^2 + A_2^3 - 2A_1^2A_2) = 0, \\ s\lambda^2 A_1 + kA_1 - pA_0A_1 - qA_1A_0^2 = 0, \\ pA_1A_2 + r\lambda A_1 + 2qA_0A_1A_2 = 0. \end{gathered}$$

We now solve the above set of equations by using the Wu elimination method [4], and obtain the following solution:

Case 1. $A_1 = 0,\ A_2 = \pm\sqrt{\frac{6s}{q}}\lambda,\ A_0 = \pm\sqrt{\frac{k-2s\lambda^2}{q}}$

$$k = 12s\lambda + r^2 \pm \frac{1}{2}rp\sqrt{\frac{6s}{q}} + \frac{sp^2}{4q}, sq > 0, q(k - 2s\lambda^2) > 0.$$

If we now take the target equation as

$$\frac{d\omega}{d\xi} = \cos\omega, \tag{14}$$

then proceeding in similar fashions, we get

Case 2. $A_0 = \pm\frac{r}{\sqrt{6qs}} - \frac{p}{2q}, A_1 = \pm\sqrt{\frac{6s}{q}}, A_2 = 0,$

$$k = \pm\frac{2pr}{\sqrt{6qs}} + \frac{r^2+36s^2}{6s} - 4s\lambda^2, qs > 0.$$

Case 3. $A_0 = \pm\frac{r}{\sqrt{6qs}}, A_1 = \pm\sqrt{\frac{3s}{2q}}\lambda, A_2 = i\sqrt{\frac{3s}{2q}}\lambda, qs > 0,$

$$k = \frac{r^2}{6s} \pm \frac{pr}{\sqrt{6qs}} - 4s\lambda^2.$$

Next, integrating $d\omega/d\xi = \sin\omega$ and taking the integration constant zero, we obtain

$$\sin\omega = \text{sech}\xi, \tag{15}$$

and

$$\cos\omega = \pm\tanh\xi. \tag{16}$$

Similarly, from (14), we get

$$\cos\omega = -\text{sech}\xi, \tag{17}$$

$$\sin\omega = \pm\tanh\xi. \tag{18}$$

According to (12), (15)-(18) and the solutions in Cases 1-3, we obtain the following solitary wave solutions of equation (1):

(I) $qs > 0$, $q(k - 2s\lambda^2) > 0$,

$$u_1(x,t) = \pm\sqrt{\frac{k-2s\lambda}{q}} \pm \sqrt{\frac{6s}{q}}\tanh\lambda(x - kt + c_0),$$

where $k = 12s\lambda^2 + r^2 \pm \frac{1}{12}rp\sqrt{\frac{6s}{q}} + \frac{sp^2}{4q}$.

(II) $qs > 0$,

$$u_2(x,t) = \pm\frac{pr}{\sqrt{6qs}} - \frac{p}{2q} \pm \sqrt{\frac{6s}{q}}\lambda\tanh\lambda(x - kt + c_0),$$

where $k = \pm\frac{2pr}{\sqrt{6qs}} + \frac{r^2+36s^2}{6s} - 4s\lambda^2$.

(III) $qs > 0$,

$$u_3(x,t) = \pm\frac{r}{\sqrt{6qs}} + \sqrt{\frac{3s}{2q}}\lambda(\pm i\tanh\xi - \text{sech}\xi)$$

where $\xi = \lambda(x - kt + c_0), k = \frac{r^2}{6s} + \frac{pr}{\sqrt{6qs}} - 4s\lambda^2$.

Note that as $|\xi| \to +\infty$, $u_3(x,t) \to \pm[\frac{r}{\sqrt{6qs}} + \sqrt{\frac{3s}{2q}}\lambda]$.

These solutions of (1) are solitary wave solutions. They are linear combinations of kink solitary and bell solitary wave solutions. They are not available in Wang [1] nor in Xia [5].

3 Travelling Wave Solutions

By measn of the same procedures described above, we may obtain solutions of (2), (3) and (4):

1. For the compound KdV equation (2), we have the following formal solitary wave solutions.

$$u_4(x,t) = -\frac{p}{2q} \pm \sqrt{\frac{6s}{q}} \tanh \lambda[x - (2s\lambda^2 + \frac{p^2}{4q})t + c_0],\ qs > 0,$$

$$u_5(x,t) = -\frac{p}{2q} \pm \sqrt{\frac{3s}{2q}} \lambda[\text{sech}\xi \pm i \tanh \xi],$$

where $\xi = \lambda[x - (\frac{1}{2}s\lambda^2 - \frac{p^2}{4q})t + c_0]$ and $qs > 0$.

2. For the KdV-Burgers equation (3), we have the following formal solitary wave solutions

$$u_6(x,t) = \pm\frac{r}{\sqrt{6qs}} \pm \sqrt{\frac{6s}{q}} \lambda \tanh[\lambda(x - \frac{12s^2\lambda^2 + r^2}{6s}t + c_0)],\ qs > 0,$$

$$u_7(x,t) = \pm[\frac{r}{\sqrt{6qs}} \pm \sqrt{\frac{3s}{2q}} \lambda[\text{sech}\xi \pm \tanh \xi],\ qs > 0,$$

where $\xi = \lambda(x - kt + c_0) = \lambda[x - \frac{r^2+3s^2\lambda^2}{6s}t + c_0]$.

3. For the mKdV equation (4), we have the the following formal solitary solutions

$$u_8(x,t) = \pm\sqrt{\frac{6s}{q}} \lambda \tanh[\lambda(x - 2s\lambda^2 t + c_0)],\ qs > 0.$$

$$u_9(x,t) = \pm\sqrt{\frac{3s}{2q}} [\tanh \lambda(x - s\lambda^2 t + c_0) - i\text{sech}\lambda(x - s\lambda^2 t + c_0)],\ qs > 0.$$

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