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Behavior of Critical Solutions of a Nonlocal Hyperbolic Problem in Ohmic Heating of Foods *

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Abstract

We study the global existence and divergence of some "critical" solutions $u^*(x,t)$ of a nonlocal hyperbolic problem modeling Ohmic heating of foods. Using comparison methods, we prove that "critical" solutions of our problem diverge globally and uniformly with respect to the space-variable as $t \to \infty$. Also, some estimates of the rate of the divergence are given.

1 Introduction

In the present work we discuss the behavior of solutions of the nonlocal hyperbolic problem

$$u_t + u_x = \frac{\lambda f(u)}{\left(\int_0^1 f(u)dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u(0,t) = 0, \quad t > 0, \tag{2}$$

$$u(x,0) = \psi(x), \quad 0 < x < 1, \tag{3}$$

at a critical value of parameter λ, say λ^* (see below), where $u = u(x,t) = u(x,t;\lambda)$ and $u^*(x,t) = u(x,t;\lambda^*)$ is referred to as a critical solution of (1-3). The function u stands for the dimensionless temperature of a moving material in a pipe (e.g. food) with negligible thermal conductivity, when an electric current flows through it; this problem occurs in the food industry (sterilization of foods), see [5] and the references therein. The parameter λ is positive and equals the square of the potential difference of the electric circuit. The nonlinear function $f(u)$ represents the dimensionless electrical resistivity of the conductor; depending upon the substance undergoing the heating, the resistivity might be an increasing, decreasing, or non-monotonic function of temperature. For most foods resistivity decreases with temperature, so we assume that $f(s)$ satisfies the condition

$$f(s) > 0, \quad f'(s) < 0, \quad s \geq 0. \tag{4}$$

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Also for simplicity, we assume that ψ is continuous (and normally, but not always, differentiable) with $\psi(0) = 0$. Although (1-3) is a hyperbolic problem, condition (4) permits us to use comparison methods, [5]. The corresponding steady-state problem to (1-3) is

$$w' = \mu f(w) > 0, \quad 0 < x < 1, \quad w(0) = 0, \tag{5}$$

with

$$\mu = \frac{\lambda}{\left(\int_0^1 f(w)dx\right)^2}. \tag{6}$$

Problem (5-6) implies

$$\mu = \mu(M) = \int_0^M \frac{ds}{f(s)} \quad \text{and} \quad \lambda = \lambda(M) = M^2 / \int_0^M \frac{ds}{f(s)}, \tag{7}$$

where $M = w(1) = \|w\|_\infty$. Also, note that $\mu(M) \geq M/f(0) \to \infty$ as $M \to \infty$, see Figure 1a. Moreover, $\lambda^* := \lim_{M\to\infty} \lambda(M) = \lim_{M\to\infty} 2Mf(M)$, by means of l'Hospital's rule.

Now if $f(s)$ is such that

$$\lambda^* = \lim_{M\to\infty} 2Mf(M) = 2c, \quad c \in (0,\infty) \quad \text{and} \quad \mu(M) > M/2f(M), \tag{8}$$

then problem (5-6) has a unique solution $w(x;\lambda)$ for each $\lambda \in (0,\lambda^*)$ (e.g. $f(s) = 1/(1+s)$), see [5]. This situation is described in Figure 1b. Relation (8) also implies that $\int_0^\infty f(s)ds = \infty$ (otherwise we would have $Mf(M) \to 0$ as $M \to \infty$, contradicting (8)).

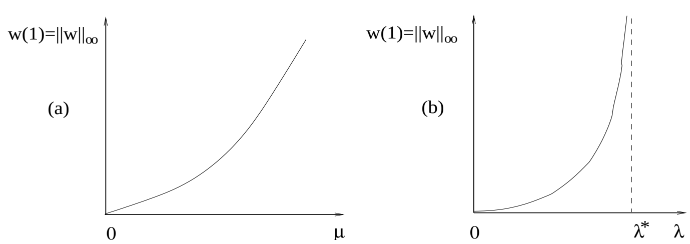

Figure 1.

It is known [5] that for $0 < \lambda < \lambda^*$, the unique steady-state solution $w(x;\lambda)$ is globally asymptotically stable and $u(x,t;\lambda)$ is global in time. Whereas, for $\lambda > \lambda^*$ the solution $u(x,t;\lambda)$ blows up in finite time. In the case where $\lambda = \lambda^*$, the only known result is that $\|u^*(\cdot,t)\|_\infty \to \infty$ as $t \to T^* \leq \infty$ (this follows by constructing a lower solution $z(x,t) = w(x;\mu(t))$ which tends to infinity as $t \to \infty$) [5]. In Section 2 we prove that $T^* = \infty$, i.e. u^* is a global in time (classical) solution which diverges ($\|u^*(\cdot,t)\|_\infty \to \infty$ as $t \to \infty$). Moreover we show that $u(x,t;\lambda^*) \to \infty$ as $t \to \infty$ for all $x \in (0,1]$ and $u_x^*(0,t) \to \infty$ as $t \to \infty$ (global divergence). In Section 3 we give some estimates of the rate of divergence of u^* and study the asymptotic form of divergence. A similar investigation, but for some nonlocal parabolic problems, is tackled in [2]; see also [3].

2 Divergence

We begin with the following result.

LEMMA 2.1. For the solutions of (5-6) there hold: (a) $w_\mu > 0$ in $(0,1]$ and (b) $w(x;\mu) \to \infty$ as $\mu \to \infty$ (or equivalently $w(x;\lambda) \to \infty$ as $\lambda \to \lambda^*-$) in $(0,1]$.

PROOF. (a) Integrating (5) over $(0,x)$ we obtain $\mu x = \int_0^{w(x)} ds/f(s)$. Differentiation of the previous relation with respect to μ gives $w_\mu = xf(w) > 0$ for $x \in (0,1]$; moreover $w_\mu(0;\mu) = 0$. (b) Integrating equation (5) again over $(0,1)$,

$$\int_0^1 f(w(x;\mu))dx = \frac{M}{\mu} = \frac{M}{\int_0^M \frac{ds}{f(s)}}, \tag{9}$$

and due to (4), (8) we obtain

$$\lim_{\mu\to\infty} \int_0^1 f(w(x;\mu))dx = \lim_{M\to\infty} \int_0^1 f(w(x;\mu(M)))dx = \lim_{M\to\infty} f(M) = 0, \tag{10}$$

which implies that $w(x;\mu) \to \infty$ as $\mu \to \infty$ (or equivalently $w(x;M) \to \infty$ as $M \to \infty$) for every $x \in (0,1]$. This proves the lemma.

PROPOSITION 2.2. Let $f(s)$ satisfy (4) and (8), then $u^*(x,t)$ is a global in time solution of (1-3) which diverges as $t \to \infty$, i.e. $\|u^*(\cdot,t)\|_\infty \to \infty$ as $t \to \infty$.

PROOF. As noted in [5], assuming $\theta(x,t) = \theta(t)$, $d\theta/dt = \lambda^*/f(\theta)$ with $\theta(0)$ large enough then $\theta(x,t)$ is an upper solution to (1-3), at $\lambda = \lambda^*$, which exists for all time, provided that $\int_0^\infty f(s)ds = \infty$. This follows immediately from $\int_{\theta(0)}^{\theta(t)} f(s)ds = \lambda^* t$, since as denoted above, (8) implies that $\int_0^\infty f(s)ds = \infty$. Recalling now that $\|u^*(\cdot,t)\|_\infty \to \infty$ as $t \to T^* \le \infty$, we finally obtain $\|u^*(\cdot,t)\|_\infty \to \infty$ as $t \to \infty$.

We now prove that $u^*(x,t)$ diverges globally.

PROPOSITION 2.3. Let $f(s)$ satisfy the hypotheses of Proposition 2.2 , then the unbounded solution $u^*(x,t)$ of (1-3) diverges globally, meaning that $u^*(x,t) \to \infty$ as $t \to \infty$ for every $x \in (0,1]$ and $u^*_x(0,t) \to \infty$ as $t \to \infty$.

PROOF. Note that there holds $(\int_0^1 f(w(x;\mu))dx)^2\mu = \lambda(\mu) < \lambda^*$ for every $\mu > 0$, since $\lambda^* = \sup\{\lambda(\mu) : \mu > 0\}$ and in addition there is no steady-state at $\lambda = \lambda^*$. Therefore we can construct a lower solution $z(x,t)$ to (1-3) at $\lambda = \lambda^*$ of the form $w(x;\mu(t))$, where $\mu(t)$ satisfies

$$\dot{\mu}(t) = \inf_{(0,1)} \left\{ \frac{f(w)}{w_\mu} \right\} \frac{(\lambda^* - \lambda(\mu))}{\left(\int_0^1 f(w)dx\right)^2} > 0, \quad t > 0, \tag{11}$$

see [5]. Equation (11) has a unique solution $\mu(t)$ which exists for all $t > 0$, [1]. Moreover, since problem (5-6) has no solution at λ^*, the unique solution $\mu(t)$ to (11) is unbounded, hence $\mu(t) \to \infty$ as $t \to \infty$. So due to Lemma 2.1, $z(x,t) = w(x;\mu(t)) \to \infty$ as $t \to \infty$ for every $x \in (0,1]$. Finally we conclude that $u^*(x,t) \to \infty$ for any $x \in (0,1]$ and $u^*_x(0,t) \ge z_x(0,t) = \mu(t)f(0) \to \infty$ as $t \to \infty$.

3 Asymptotic form of divergence

In this section, using similar ideas as in the case of blow-up for a parabolic problem, [3, 4], we obtain the asymptotic form of divergence. First, we construct a special upper solution of (1-3) giving a useful upper estimate of the rate of divergence of $u^*(x,t)$ (this upper solution is global in time and can serve as an alternative way to prove Proposition 2.2). Therefore we seek a prospective upper solution $V(x,t)$ of the form:

$$V(x,t) = w(y(x);\mu(t)), \quad 0 \le x \le \varepsilon, \quad t > 0, \tag{12}$$

$$V(x,t) = M(t) = \max_{0\le x\le\varepsilon} w(y(x);\mu(t)), \quad \varepsilon < x \le 1, \quad t > 0, \tag{13}$$

where $0 < y(x) = x/\varepsilon < 1$ (ε is a constant in $(0,1)$) and $w(y(x);\mu(t))$ satisfies the problem

$$w_x = \frac{\mu(t)}{\varepsilon} f(w), \quad 0 < x < \varepsilon, \quad w(0) = 0. \tag{14}$$

It is obvious from the definition of $V(x,t)$ that V is continuous at $x = \varepsilon$ and $V(0,t) = 0$. Due to Lemma 2.1 we have that $w_\mu(x;\mu) = w_\nu(x;\nu)/\varepsilon \ge 0$ for $0 \le x \le 1$, where $\nu = \mu/\varepsilon$. Hence, by choosing a sufficiently large $\mu(0)$, $V(x,0) = w(\psi(x);\mu(0)) \ge \psi(x)$ for $0 \le x \le 1$. Moreover

$$\int_0^1 f(V)dx = (1-\epsilon) f(M) + \frac{\varepsilon}{\mu}\int_0^\varepsilon w_x\, dx = (1-\varepsilon) f(M) + \frac{\varepsilon M}{\mu}. \tag{15}$$

Also (7) implies that

$$\mu(M) f(M) \le M, \tag{16}$$

and since $\lim_{M\to\infty} M f(M) = c > 0$, we get

$$f(M) \sim \frac{c}{M} \quad \text{and} \quad \frac{M^2}{\mu(M)} \sim 2c \quad \text{as} \quad M \to \infty. \tag{17}$$

Finally (17) implies

$$\sqrt{\mu(M)} f(M) \sim \sqrt{\frac{c}{2}} \quad \text{as} \quad M \to \infty. \tag{18}$$

For $0 \le x \le \varepsilon$,

$$\begin{aligned} G(V) &\equiv V_t + V_x - \frac{\lambda^* f(V)}{\left(\int_0^1 f(V)dx\right)^2} \\ &= w_\mu \dot{\mu}(t) + \frac{\mu(t) f(w)}{\varepsilon} - \frac{2c f(w)}{\left[(1-\varepsilon) f(M) + \frac{\varepsilon}{\mu} M\right]^2} \\ &\sim w_\mu \dot{\mu}(t) + \frac{\mu(t) f(w)}{\varepsilon}\left[1 - 1/\left(\frac{1-\varepsilon}{2\sqrt{\varepsilon}} + \sqrt{\varepsilon}\right)^2\right], \quad M \gg 1, \end{aligned}$$

due to (15), (17) and (18). We note that

$$\frac{1-\varepsilon}{2\sqrt{\varepsilon}} + \sqrt{\varepsilon} = \frac{\varepsilon+1}{2\sqrt{\varepsilon}} > 1, \text{ for any } 0 < \varepsilon < 1, \tag{19}$$

thus $G(V) \gtrsim w_\mu \dot{\mu}(t) > 0$ for $x \in [0, \epsilon]$, since $w_\mu > 0$ in $(0, 1]$ and provided that $\dot{\mu}(t) > 0$ (see below). For $\varepsilon < x \leq 1$ we obtain

$$\begin{aligned} G(V) &= \dot{M}(t) - \frac{2cf(M)}{\left[(1-\varepsilon)f(M) + \frac{\varepsilon}{\mu}M\right]^2} \\ &\sim \dot{M}(t) - \frac{\mu(M)f(M)}{\varepsilon\left[\frac{1-\varepsilon}{2\sqrt{\varepsilon}} + \sqrt{\varepsilon}\right]^2} \gtrsim \dot{M}(t) - \frac{\mu(M)f(M)}{\varepsilon}, \quad M \gg 1, \end{aligned}$$

using (17), (18) and (19). Now by choosing $M(t)$ such that

$$\dot{M}(t) = \frac{\mu(M)f(M)}{\varepsilon} > 0, \quad t > 0, \tag{20}$$

we finally take $G(V) \gtrsim 0$ for $\varepsilon < x \leq 1$ and $M \gg 1$. Equation (20) implies that $M(t)$ is increasing, so $\dot{\mu}(t) = \dot{M}(t)/\frac{dM}{d\mu} > 0$. Also integrating (20) and using estimate (16), we get

$$\frac{t}{\varepsilon} = \int_{M(0)}^{M(t)} \frac{ds}{\mu(s)f(s)} \geq \int_{M(0)}^{M(t)} \frac{ds}{s} = \ln M(t) - \ln M(0). \tag{21}$$

This relation implies that if $M(t) \to \infty$ then $t \to \infty$. Whence taking $M(0) \gg 1$ we get that $V(x, t)$ is an upper solution to (1-3) at $\lambda = \lambda^*$, which exists for all time.

Now, from (21), we get that $\|u^*(\cdot, t)\|_\infty$ does not tend to infinity faster than $M(0)e^{t/\varepsilon}$ does as $t \to \infty$ for any $0 < \varepsilon < 1$, that is, $N(t) \lesssim M(0)e^{t/\varepsilon}$ as $t \to \infty$, where $N(t) = \|u^*(\cdot, t)\|_\infty$. Before giving a lower estimate of the rate of divergence of $u^*(x, t)$, we prove the following:

PROPOSITION 3.1. The divergence of $u^*(x, t)$ is uniform on compact subsets of $(0, 1]$, meaning that $\lim_{t\to\infty} |u^*(x_1, t) - u^*(x_2, t)| = 0, \quad 0 < \delta \leq x_1 < x_2 \leq 1$, for any positive δ.

PROOF. Using the variable $y = x - t$ in place of x, equation (1), at $\lambda = \lambda^*$, can be written as

$$dU^*/dt = g(t)f(U^*), \tag{22}$$

where $U^*(y, t) = u^*(x, t)$ and $g(t) = \lambda^*/(\int_{-t}^{1-t} f(U^*)dy)^2$. Since (4) holds, (22) implies $dU^*/dt \geq g(t)f(N) = dN/dt$, where $N(t) = \max_y U^*(y, t)$. Integrating the last inequality we obtain $U^*(y, t) - U^*(y, 0) \geq N(t) - N(0)$, which implies that $N(t) \geq U^*(y, t) = u^*(x, t) \gtrsim N(t)$ as $t \to \infty$ or $u^*(x, t) \sim N(t)$ as $t \to \infty$ for every $x \in (0, 1]$, since $u^*(x, t)$ diverges globally. Thus $|u^*(x_1, t) - u^*(x_2, t)| \leq (N(t) - u^*(x_2, t)) \to 0$ as $t \to \infty$, for $0 < \delta \leq x_1 < x_2 \leq 1$. The proof is complete.

From relation (4) we have that $N(t)$ satisfies $dN/dt = \lambda^* f(N)/(\int_0^1 f(u^*)dx)^2 \geq \lambda^* f(N)/f^2(0)$. Using (17) we take $dN/dt \gtrsim \lambda^* c/Nf^2(0)$ as $t \to \infty$ or equivalently

$N^2(t)/2 - N^2(t_1)/2 \gtrsim \lambda^* c/f^2(0)(t-t_1)$ for $t > t_1 \gg 1$. Finally we obtain $N(t) \gtrsim \frac{\lambda^*}{f(0)}\sqrt{t}$ as $t \to \infty$, since $\lambda^* = 2c$.

Thus we have proved:

PROPOSITION 3.2. Let f satisfy the hypotheses of Proposition 2.2, then $u^*(x,t)$ grows at least as the square root of time t ($\|u^*(\cdot,t)\|_\infty \gtrsim C\sqrt{t}$, $C = \lambda^*/f(0)$) as $t \to \infty$ but no faster than exponentially ($\|u^*(\cdot,t)\|_\infty \lesssim M(0)e^{t/\varepsilon}$, for any $0 < \varepsilon < 1$) as $t \to \infty$.

It can be expected, due to Proposition 3.1, that for $t \gg 1$, $u^* \sim N$ i.e. $u^*(x,t)$ exhibits a flat divergence profile, except for a boundary layer whose thickness vanishes as $t \to \infty$ (by the boundary layer, we mean the region near to $x = 0$ where the solution $u^*(x,t)$ follows a fast transition between the divergence regime and the assigned zero boundary condition). Therefore in the main core region we neglect u_x^* so

$$dN/dt \sim g(t)f(N) \text{ as } t \to \infty, \text{ where } g(t) = \frac{\lambda^*}{\left(\int_0^1 f(u^*)dx\right)^2}.$$

Significant contributions to the integral $\int_0^1 f(u^*)dx$ can come from the largest core (region) which has width ~ 1 and its contribution is $\sim f(N)$) and from the boundary layer where $f(u^*)$ is larger, since f is decreasing and $u^* < N$; $f(u^*)$ is $O(1)$ and $f(u^*) \geq k > 0$ wherever u^* is $O(1)$. If the boundary layer has width $\delta = \delta(t)$ then

$$\sqrt{\frac{\lambda^*}{g(t)}} = O(\delta(t)) + O(f(N(t))), \quad t \gg 1,$$

and either $g(t) = O(\delta^{-2}(t))$ or $g(t) = O(f^{-2}(N(t)))$, whichever is the larger for $t \gg 1$.

Supposing that $\delta(t) \ll f(N(t))$ as $t \to \infty$ then the core dominates and $g(t) \sim \lambda^*/f^2(N(t))$ for $t \to \infty$. Hence

$$dN/dt \sim \frac{\lambda^*}{f(N)} \quad \text{for } t \to \infty,$$

and using (17) we finally obtain $N(t) \sim N(0)e^{2t}$ as $t \to \infty$, which contradicts the fact that $N(t) \lesssim M(0)e^{t/\varepsilon}$ as $t \to \infty$, for any $0 < \varepsilon < 1$ (see Proposition 3.2). Also assuming that $\delta(t) = O(f(N(t)))$ as $t \to \infty$ we arrive at a contradiction as before. There remains only one possibility: $\delta(t) \gg f(N(t))$ as $t \to \infty$.

Thus the boundary layer has width $\delta(t) = O(g(t)^{-1/2}) \gg f(N(t))$, as $t \to \infty$; using now (17) and taking into account Proposition 3.2, we obtain

$$\delta(t) \gtrsim \frac{c}{M(0)} e^{-t/\varepsilon} \text{ as } t \to \infty, \text{ for every } 0 < \varepsilon < 1,$$

i.e. the width of the boundary layer decreases no faster than exponentially. In the boundary layer, u^* is $O(1)$ and u_t^* is negligible compared to u_x^* (due to the continuity of u_t^*, u_x^* we get $|u_t^*(x,t)| < \epsilon$, $0 < x < \delta(t)$, $t > 0$, for every $\epsilon > 0$, and $u_x^*(0,t) - \epsilon < u_x^*(x,t) \to \infty$, $0 < x < \delta(t)$, as $t \to \infty$, since $u_x^*(0,t) \to \infty$ as $t \to \infty$). There has to be a balance between u_x^* and $g(t)f(u^*)$, i.e.

$$u_x^* \sim g(t)f(u^*), \text{ for } 0 < x < \delta(t), \text{ as } t \to \infty. \tag{23}$$

So in the boundary layer $u^*(x,t)$ behaves like $w(x;\mu(t))$ as $t \to \infty$ (this fact justifies the form of upper solution $V(x,t)$ constructed above).

From the above analysis and (23), we obtain

$$u_x^*(x,t) \sim \frac{f(u^*)}{f^2(0)\delta^2(t)}, \text{ for } 0 < x < \delta(t), \text{ as } t \to \infty. \tag{24}$$

Integrating the last relation over $(0,x)$ and using (17) we obtain that

$$u^*(x,t) \sim \frac{\sqrt{\lambda^*\, x}}{f(0)\delta(t)} \quad \text{for } t \to \infty, \tag{25}$$

as we leave the boundary $x = 0$. Leaving the boundary layer, relation (25) becomes $N(t) \sim \sqrt{\lambda^*}/\sqrt{f^2(0)\delta(t)}$ as $t \to \infty$, and using Proposition 3.2, we get

$$\delta(t) \lesssim \frac{1}{\lambda^*}\, t^{-1} \text{ as } t \to \infty. \tag{26}$$

Estimate (26) implies that the size (width) of the boundary layer decreases faster than t^{-1} as $t \to \infty$, which is the analogous result to the one holding in the case of blow-up for nonlocal diffusion equations, see [4, 6].

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