

# Optimal Control for Third Degree Polynomial Systems \*

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## Abstract

This paper presents the optimal regulator for a nonlinear system state given by a polynomial equation of degree 3 with linear control input and quadratic cost criterion. The optimal regulator equations are obtained using the duality principle, which is applied to the optimal filter for a polynomial system state of degree 3 over linear observations. The obtained results are applied to solution of the optimal control problem for a nonlinear automotive system. Simulation results are compared for the optimal polynomial regulator given in this paper and the linear optimal regulator.

## 1 Introduction

Although the optimal control (regulator) problem as well as the filtering one were solved in the 1960s [4, 3], the optimal control function for nonlinear systems has to be determined by using the general principles of maximum [7] or dynamic programming [2] which do not provide an explicit form for the optimal control in most cases. However, taking into account that the optimal control problem can be solved in the linear case by applying the duality principle to the solution of the optimal filtering problem, this paper exploits the same idea for designing the optimal control in a polynomial system with linear control input, using the optimal filter for polynomial system states over linear observations. Based on the obtained polynomial filter of the third degree [1], the optimal regulator for a polynomial system of degree 3 with linear control input and quadratic cost criterion is obtained in a closed form, finding the optimal regulator gain matrix as dual transpose to the optimal filter gain one and constructing the optimal regulator gain equation as dual to the variance equation in the optimal filter. The results obtained by virtue of the duality principle could be rigorously verified through the general equations of [7] or [2] applied to a specific polynomial case, although the physical duality seems obvious: if the optimal filter exists in a closed form, the optimal closed-form regulator should also exist, and vice versa. Finally, the obtained optimal control for a polynomial system of the third degree is applied to regulation

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of a nonlinear automotive system [5] whose state equation for car orientation angle is nonlinear (contains tangent). To apply the polynomial regulator, the nonlinear equation is expanded into its Taylor polynomial up to degree 3. The optimal regulator equations for a polynomial state of third degree are written and then compared to the optimal linear regulator for the linearized system. Simulations are conducted for both polynomial and linear regulators applied to the original nonlinear system. The simulation results show significant advantage of the polynomial regulator in comparison to the linear one, five times in the values of the controlled variable and ten times in the criterion performance.

This relatively simple case treated in the paper seems to be important for practical applications, since a nonlinear state equation can usually be well approximated by a polynomial of degree 3 and the control input is, as a rule, linear. Moreover, the optimal control problem for a polynomial state equation of lower degree is significant itself, because many, for example, chemical processes are described by quadratic equations (see [6]). The quadratic state equation is, of course, a particular case of the third degree one, as well the cubic state equation is a particular case of that of fourth degree, etc.

The paper is organized as follows. Section 2 states the optimal control problem for a polynomial system of degree 3 and the duality principle for a closed-form situation. For reference purposes, the optimal filtering equations for a polynomial state equation of degree 3 and linear observations are briefly recalled in Section 3. The optimal control problem for a polynomial system state of degree 3 is solved in Section 4. Section 5 presents application of the optimal polynomial regulator to a nonlinear automotive system with two state variables, orientation and steering angles, with the objective to increase the value of the orientation angle and consume the minimum control energy. Graphic simulation results are conducted for polynomial control of degree 3 and compared with those for linear control.

## 2 Optimal Control Problem

Consider the polynomial system

$$dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + G(t)u(t)dt, \quad x(t_0) = x_0, \quad (1)$$

where  $x(t) \in R^n$  is the system state,  $x^2(t) = (x_1^2(t), \dots, x_n^2(t))$ ,  $x^3(t) = (x_1^3(t), \dots, x_n^3(t))$ , and  $u(t)$  is the control variable. The quadratic cost function to be minimized is defined as follows:

$$J = \frac{1}{2}[x(T) - x_1]^T \psi [x(T) - x_1] + \frac{1}{2} \int_{t_0}^T (u^T(s)R(s)u(s) + x^T(s)L(s)x(s)) ds, \quad (2)$$

where  $x_1$  is a given vector,  $\psi$ ,  $R$ ,  $L$  are positive (nonnegative) definite symmetric matrices, and  $T > t_0$  is a certain time moment. We remark that the transpose of a vector  $x$  is also denoted by  $x^T$ , which, however, should not cause any confusion.

The optimal control problem is to find the control  $u(t)$ ,  $t \in [t_0, T]$ , that minimizes the criterion  $J$  along with the trajectory  $x^*(t)$ ,  $t \in [t_0, T]$ , generated upon substituting  $u^*(t)$  into the state equation (1). To find the solution to this optimal control problem,

the duality principle [4] could be used. For linear systems, if the optimal control exists in the optimal control problem for a linear system with the quadratic cost function  $J$ , the optimal filter exists for the dual linear system with Gaussian disturbances and can be found from the optimal control problem solution, using simple algebraic transformations (duality between the gain matrices and between the gain matrix and variance equations), and vice versa. Taking into account the physical duality of the filtering and control problems, the last conjecture should be valid for all cases where the optimal control (or, vice versa, the optimal filter) exists in a closed finite-dimensional form. This proposition is now applied to a third order polynomial system, for which the optimal filter has already been obtained (see [1]).

### 3 Optimal Filter

In this section, the optimal filtering equations for a polynomial state equation of degree 3 over linear observations (obtained in [1]) are briefly recalled for reference purposes. Let an unobservable random process  $x(t)$  satisfy a polynomial equation of third degree

$$dx(t) = (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt + b(t)dW_1(t), \quad x(t_0) = x_0, \quad (3)$$

and linear observations are given by:

$$dy(t) = (A_0(t) + A(t)x(t))dt + B(t)dW_2(t),$$

where  $x \in R^n$ ,  $x^2(t) = (x_1^2(t), \dots, x_n^2(t))$ ,  $x^3(t) = (x_1^3(t), \dots, x_n^3(t))$ .  $W_1(t)$  and  $W_2(t)$  are Wiener processes, whose weak derivatives are Gaussian white noises and which are assumed independent of each other and of the Gaussian initial value  $x_0$ .

The filtering problem is to find dynamical equations for the best estimate for the real process  $x(t)$  at time  $t$ , based on the observations  $Y(t) = [y(s) \mid t_0 \leq s \leq t]$ , that is the conditional expectation  $m(t) = E[x(t) \mid Y(t)]$  of the real process  $x(t)$  with respect to the observations  $Y(t)$ . Let  $P(t) = E[(x(t) - m(t))(x(t) - m(t))^T \mid Y(t)]$  be the estimate covariance (correlation function).

The following notations are used. Let  $m(t) = (m_1(t), \dots, m_n(t)) \in R^n$  be the best estimate vector;  $P(t) \in R^{n \times n}$  be the covariance matrix;  $p(t) \in R^n$  be the vector whose components are the variances of the components of  $x(t)$ , i.e., the diagonal elements of  $P(t)$ ;  $m^2(t) = (m_1^2(t), \dots, m_n^2(t))$ ;  $m^3(t) = (m_1^3(t), \dots, m_n^3(t))$ ;  $P(t)m(t)$  be the conventional product of a matrix  $P(t)$  by a vector  $m(t)$ ; and  $p(t) * m(t)$  be the product of two vectors defined componentwise:  $p(t) * m(t) = [p_1(t)m_1(t), \dots, p_n(t)m_n(t)]$ .

The solution to the stated problem is given by the following system of filtering equations, which is closed with respect to the introduced variables,  $m(t)$  and  $P(t)$ :

$$\begin{aligned} & dm(t) \\ = & (a_0(t) + a_1(t)m(t) + a_2(t)p(t) + a_2(t)m^2(t) + a_3(t)(3p(t) * m(t) + m^3(t)))dt \\ & + P(t)A^T(t)(B(t)B^T(t))^{-1}(dy - (A_0(t) + A(t)m(t))dt), \end{aligned} \quad (4)$$

$$m(t_0) = E[x(t_0)/y(t_0)].$$

$$\begin{aligned}
dP(t) = & (a_1(t)P(t) + P(t)a_1^T(t) + 2a_2(t)m(t) * P(t) \\
& + 2(P(t) * m^T(t))a_2^T(t) + 3a_3(t)(p(t) * P(t)) + 3(p(t) * P(t))^T a_3^T(t) \\
& + 3a_3(t)(m^2(t) * P(t)) + 3(P(t) * (m^2(t))^T)a_3^T(t) + b(t)b^T(t) \\
& - P(t)A^T(t)(B(t)B^T(t))^{-1}A(t)P(t))dt, \tag{5}
\end{aligned}$$

$$P(t_0) = E((x(t_0) - m(t_0))(x(t_0) - m(t_0))^T / y(t_0)),$$

where the product  $m(t) * P(t)$  between a vector  $m(t)$  and a matrix  $P(t)$  is defined as the matrix whose rows are equal to rows of  $P(t)$  multiplied by the same corresponding element of  $m(t)$ :

$$\begin{aligned}
& \begin{bmatrix} m_1(t) & P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ m_2(t) & P_{21}(t) & P_{22}(t) & \cdots & P_{2n}(t) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_n(t) & P_{n1}(t) & P_{n2}(t) & \cdots & P_{nn}(t) \end{bmatrix} \\
= & \begin{bmatrix} m_1(t)P_{11}(t) & m_1(t)P_{12}(t) & \cdots & m_1(t)P_{1n}(t) \\ m_2(t)P_{21}(t) & m_2(t)P_{22}(t) & \cdots & m_2(t)P_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ m_n(t)P_{n1}(t) & m_n(t)P_{n2}(t) & \cdots & m_n(t)P_{nn}(t) \end{bmatrix}.
\end{aligned}$$

The transposed matrix  $P(t) * m^T(t) = (m(t) * P(t))^T$  is defined as the matrix whose columns are equal to columns of  $P(t)$  multiplied by the same corresponding element of  $m(t)$ :

$$\begin{aligned}
& \begin{bmatrix} m_1(t) & m_2(t) & \cdots & m_n(t) \end{bmatrix} \begin{bmatrix} P_{11}(t) & P_{12}(t) & \cdots & P_{1n}(t) \\ P_{21}(t) & P_{22}(t) & \cdots & P_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1}(t) & P_{n2}(t) & \cdots & P_{nn}(t) \end{bmatrix} \\
= & \begin{bmatrix} m_1(t)P_{11}(t) & m_2(t)P_{12}(t) & \cdots & m_n(t)P_{1n}(t) \\ m_1(t)P_{21}(t) & m_2(t)P_{22}(t) & \cdots & m_n(t)P_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ m_1(t)P_{n1}(t) & m_2(t)P_{n2}(t) & \cdots & m_n(t)P_{nn}(t) \end{bmatrix}.
\end{aligned}$$

Thus, the equation (4) for the optimal estimate  $m(t)$  and the equation (5) for its covariance matrix  $P(t)$  form a closed system of filtering equations in the case of a polynomial state equation of degree 3 and linear observations.

## 4 Optimal Solution

Let us return to the optimal control problem for the polynomial state (1) with linear control input and the cost function (2). This problem is dual to the filtering problem for the polynomial state (3) of degree 3 and linear observations. Since the optimal polynomial filter gain matrix in (4) is equal to

$$K_f = P(t)A^T(t)(B(t)B^T(t))^{-1},$$

the gain matrix in the optimal control problem takes the form of its dual transpose

$$K_c = (R(t))^{-1}G^T(t)Q(t),$$

and the optimal control law is given by

$$u^*(t) = K_c x = (R(t))^{-1}G^T(t)Q(t)x(t), \quad (6)$$

where the matrix function  $Q(t)$  is the solution of the following equation dual to the variance equation (5)

$$\begin{aligned} dQ(t) = & (-a_1^T(t)Q(t) - Q(t)a_1(t) - 2a_2^T(t)Q(t) * x^T(t) - 2x(t) * Q(t)a_2(t) \\ & - 3a_3^T(t)Q(t) * q^T(t) - 3q(t) * Q(t)a_3(t) - 3a_3^T(t)Q(t) * ((x^2(t))^T) \\ & - 3(x^2(t) * Q(t))a_3(t) + L(t) - Q(t)G(t)R^{-1}(t)G^T(t)Q(t))dt, \end{aligned} \quad (7)$$

with the terminal condition  $Q(T) = \psi$ . The binary operation  $*$  has been introduced in Section 3, and  $q(t) = (q_1(t), q_2(t), \dots, q_n(t))$  denotes the vector consisting of the diagonal elements of  $Q(t)$ .

Upon substituting the optimal control (6) into the state equation (1), the optimally controlled state equation is obtained

$$\begin{aligned} dx(t) &= (a_0(t) + a_1(t)x(t) + a_2(t)x^2(t) + a_3(t)x^3(t))dt \\ &\quad + G(t)(R(t))^{-1}G^T(t)Q(t)x(t)dt, \\ x(t_0) &= x_0, \end{aligned}$$

Note that if the real state vector  $x(t)$  is unknown (unobservable), the optimal controller uniting the obtained optimal filter and regulator equations, can be constructed using the separation principle [4] for polynomial systems, which should also be valid if solutions of the optimal filtering and control problems exist in a closed finite-dimensional form.

The results obtained in this section by virtue of the duality principle could be rigorously verified through the general equations of the Pontryagin maximum principle [7] or Bellman dynamic programming [2].

## 5 Application to Automotive System

This section presents application of the obtained optimal regulator for a polynomial system of degree 3 with linear control input and quadratic criterion to controlling the state variables, orientation and steering angles, in the nonlinear kinematical model of car movement [5] given by the following nonlinear equations

$$\begin{aligned} dx(t) &= v \cos(\phi(t))dt \\ dy(t) &= v \sin(\phi(t))dt \\ d\phi(t) &= (v/l) \tan(\delta(t))dt \\ d\delta(t) &= u(t)dt \end{aligned} \quad (8)$$

Here,  $x(t)$  and  $y(t)$  are Cartesian coordinates of the mass center of the car,  $\phi(t)$  is the orientation angle,  $v$  is the velocity,  $l$  is the longitude between the two axes of the car,  $\delta(t)$  is the steering wheel angle, and  $u(t)$  is the control variable (steering angular velocity).

The optimal control problem is to maximize the orientation angle  $\phi$  using the minimum energy of control  $u$ . The examined values of the velocity and longitude are  $v = 17$ ,  $l = 2$ , and the motion time is  $T = 0.1$ , which correspond to the idle engine mode of a full-size car in the time interval of 6 seconds. The initial conditions for the angles are  $\phi(0) = 0.1$  and  $\delta(0) = 0.1$ . In other words, the problem is to make the maximum turn of the running wheels from their initial position, using the minimum steering energy.

The corresponding criterion  $J$  to be minimized takes the form

$$J = \frac{1}{2}[\phi(T) - \phi^*]^2 + \frac{1}{2} \int_0^T u^2(t) dt \quad (9)$$

where  $T = 0.1$ , and  $\phi^* = 1$  is a large value of  $\phi(t)$  *a priori* unreachable for time  $T$ . To apply the obtained optimal control algorithms to the nonlinear system (8), let us make the Taylor expansion of the two last equations in (8) at the origin up to degree 3 (the fourth degree does not appear in the Taylor series for tangent)

$$\begin{aligned} d\phi(t) &= \left(\frac{v}{l}\right) \delta(t) dt + \left(\frac{v}{l}\right) \left(\frac{\delta^3(t)}{3}\right) dt \\ d\delta(t) &= u(t) dt \end{aligned} \quad (10)$$

Now, since  $R = 1$  and  $G^T = [0, 1]$  in (9), the optimal control law (6) takes the form  $u^*(t) = q_{21}(t)\phi(t) + q_{22}(t)\delta(t)$ , where the elements  $q_{11}(t)$ ,  $q_{21}(t)$ ,  $q_{22}(t)$  of the symmetric matrix  $Q(t)$  satisfy the equations

$$\begin{aligned} dq_{11}(t) &= -q_{21}^2(t) \\ dq_{12}(t) &= -\frac{3v}{l}q_{11}^2(t) - q_{12}(t)q_{22}(t) - \frac{v}{l}q_{11}(t) - \frac{3v}{l}\phi^2 q_{11}(t) \\ dq_{22}(t) &= -\frac{2v}{l}q_{12}(t) - \frac{6v}{l}q_{12}(t)q_{22}(t) - \frac{6v}{l}\delta^2 q_{12}(t) - q_{22}^2(t) \end{aligned} \quad (11)$$

The system composed of the two last equations of (8) and the equations (10) should be solved with initial conditions  $\phi(0) = 0.1$ ,  $\delta(0) = 0.1$  and terminal conditions  $q_{11}(T) = 1$ ,  $q_{12}(T) = 0$ ,  $q_{22}(T) = 0$ . This boundary problem is solved numerically using the iterative method of direct and reverse passing as follows. The first initial conditions for  $q$ 's are guessed, and the system is solved in direct time with the initial conditions at  $t = 0$ , thus obtaining certain values for  $\phi$  and  $\delta$  at the terminal point  $T = 0.1$ . Then, the system is solved in reverse time, taking the obtained terminal values for  $\phi$  and  $\delta$  in direct time as the initial values in reverse time, thus obtaining certain values for  $q$ 's at the initial point  $t = 0$ , which are taken as the initial values for the passing in direct time, and so on. The given initial conditions  $\phi(0) = 0.1$ ,  $\delta(0) = 0.1$  are kept fixed for any direct passing, and the given terminal conditions  $q_{11}(T) = 1$ ,  $q_{12}(T) = 0$ ,  $q_{22}(T) = 0$  are used as the fixed initial conditions for any reverse passing. The algorithm stops when the system arrives at values  $q_{11}(T) = 1$ ,  $q_{12}(T) = 0$ ,  $q_{22}(T) = 0$  after direct passing and at values  $\phi(0) = 0.1$ ,  $\delta(0) = 0.1$  after reverse passing. The initial conditions for  $q$ 's in the final direct iteration are  $q_{11}(0) = 1.32$ ,  $q_{12}(0) = 16$ ,  $q_{22}(0) = 1640$ . The obtained simulation graphs for  $\phi$  and

the criterion  $J$  are given in Fig. 2. These results for polynomial regulator of degree 3 are then compared to the results obtained using the optimal linear regulator, whose matrix  $Q(t)$  elements satisfy the Riccati equations

$$\begin{aligned} dq_{11}(t) &= -q_{12}^2(t) \\ dq_{12}(t) &= -q_{12}q_{22} - \frac{v}{T}q_{11} \\ dq_{22}(t) &= -\frac{2v}{T}q_{12} - q_{22}^2 \end{aligned} \quad (12)$$

with terminal conditions  $q_{11}(T) = 1, q_{12}(T) = 0, q_{22}(T) = 0$ . Note that in the linear case the only reverse passing for  $q$ 's is necessary, because the system (12) does not depend on  $\phi$  and  $\delta$ , and the initial values for  $q$ 's at  $t = 0$  are obtained after single reverse passing. The initial conditions for  $q$ 's in the direct iteration are  $q_{11}(0) = 1.025, q_{12}(0) = 0.87, q_{22}(0) = 0.74$ . The simulation graphs for the linear case are given in Figure 1, which consists of the graph of the variable  $\phi$  satisfying the original system (8) and controlled using the optimal linear regulator defined by (12) and the graph of the corresponding values of the criterion  $J$ .

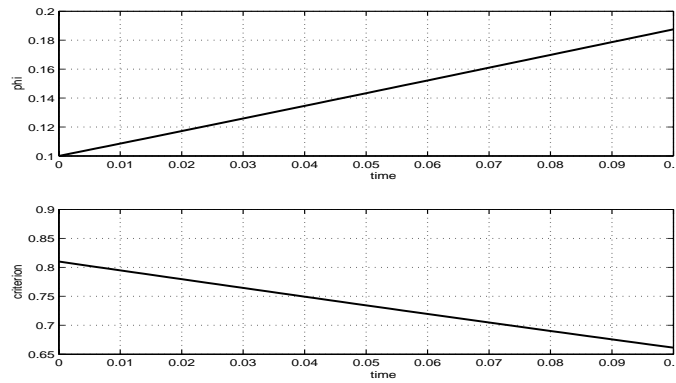


Figure 1.

In Figure 2, we show the graph of the variable  $\phi$  satisfying the original system (8) and controlled using the optimal third order polynomial regulator defined by (11) and the graph of the corresponding values of the criterion  $J$ .

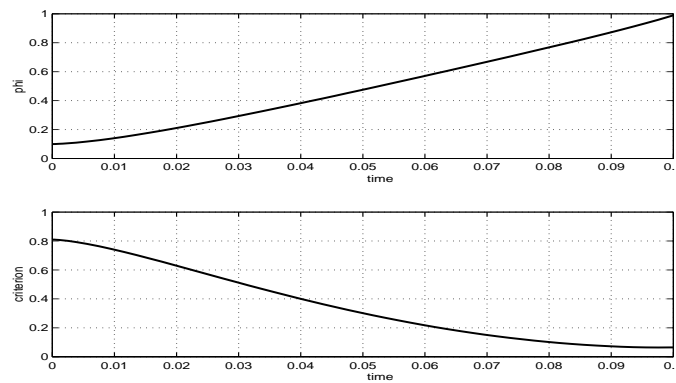


Figure 2.

The obtained values of the controlled variable  $\phi$  and the criterion  $J$  are compared for the optimal third order polynomial and linear regulators at the terminal time  $T = 0.1$  in the following table (corresponding to Figs. 1 and 2).

Linear regulator	Third degree polynomial regulator
$\phi(0.1) = 0.1875$	$\phi(0.1) = 0.989$
$J = 0.661$	$J = 0.065$

Graphs of control functions  $u^*(t)$  corresponding to the optimal linear regulator and the optimal third order polynomial regulator are given in Figs. 3 and 4, respectively.

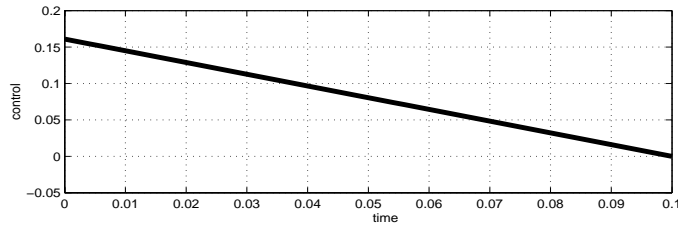


Figure 3.

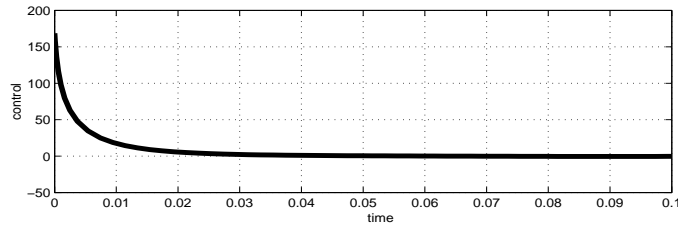


Figure 4.

## 6 Conclusions

The simulation results show that the values of the controlled variable  $\phi$  at the terminal point  $T = 0.1$  are five times greater for the third order regulator than for the linear one and the criterion value at the terminal point is ten times less for the third order regulator. Thus, the third order polynomial regulator controls the system variables significantly better than the linear one from both points of view. The obtained results show that the best gain matrix based on the linearized model could still be too far from achieving the optimal performance. The considered example validates design and implementation of the regulators based on polynomial approximations of nonlinear systems.

Finally note that the better performance of the cost function and controlled variable has been achieved without changing the system dynamics (in both simulation cases, the designed control algorithms are applied to the original nonlinear system (8)), but by assigning a better regulator gain matrix ( $Q(t)$  satisfies (11) instead of (12)). Thus, the principal result in the considered application consists in designing a better regulator and not in using more accurate system dynamics, as it could seem after the first glance.



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