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Explicit Asymptotic Stability Criteria for Neutral Differential Equations with Two Delays *

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Abstract

Explicit necessary and sufficient conditions are derived for the asymptotic stability of a neutral differential equation with two delays.

1 Introduction

Consider the following equation

$$x'(t) + cx'(t-2\tau) + ax(t) + ax(t-\tau) = 0, \tag{1}$$

where a, c and τ are constants, $c \neq 0$ and $\tau > 0$. The characteristic equation of equation (1) is

$$h(\lambda) := \lambda(1 + ce^{-2\tau\lambda}) + a + ae^{-\tau\lambda} = 0. \tag{2}$$

In this paper, we establish explicit necessary and sufficient conditions for asymptotic stability of equation (1), and show that the trivial solution of equation (1) is asymptotically stable if, and only if, it is exponentially asymptotically stable.

THEOREM 1.1. The trivial solution of equation (1) is asymptotically stable if, and only if, all the roots of equation (2) possess negative real parts. Furthermore, the trivial solution of equation (1) is asymptotically stable if, and only if, it is exponentially asymptotically stable.

THEOREM 1.2. The trivial solution of equation (1) is asymptotically stable if, and only if, $a > 0$ and $0 < |c| < 1$, and either one of the following sets of conditions hold:

$$-1 < c \leq \frac{1}{3},\ c \neq 0. \tag{3}$$

$$\frac{1}{3} < c < 1,\ 0 < \tau < \frac{1-c}{a}\sqrt{\frac{1+c}{3c-1}}\arccos\frac{1-c}{2c}. \tag{4}$$

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2 Preparatory Lemmas

Consider the function

$$F(z) = f(z, \cos z, \sin z) \tag{5}$$

where $f(z,u,v)$ is a polynomial with real coefficients in z,u,v. Let $z^r\psi_r^{(s)}(u,v)$ denote the principal term of $f(z,u,v)$, let $z^r\psi_*^{(s)}(u,v)$ be its head term, and set $\phi_*^{(s)}(z) = \psi_*^{(s)}(\cos z, \sin z)$.

LEMMA 2.1 ([3, Theorem A.4]). Let $f(z,u,v)$ be a polynomial with principal term $z^r\psi_r^{(s)}(u,v)$. If ε is such that $\phi_*^{(s)}(\varepsilon + iy) \neq 0$ for $y \in R$, then for sufficiently large integer k, the function $F(z)$ defined in (5) will have exactly $4ks + r$ zeros in the strip $-2k\pi + \varepsilon \leq \operatorname{Re} z < 2k\pi + \varepsilon$. Consequently, the function $F(z)$ will have only real zeros if, and only if, for sufficiently large integer k, it has exactly $4ks + r$ real zeros in the strip $-2k\pi + \varepsilon \leq \operatorname{Re} z < 2k\pi + \varepsilon$.

Consider the function

$$H(z) = h(z, e^z), \tag{6}$$

where $h(z,w)$ is a polynomial in z, w. Let

$$F(y) = \text{Re } H(iy),\ G(y) = \text{Im } H(iy). \tag{7}$$

LEMMA 2.2 ([3, Theorem A.3]). Let $h(z,w)$ be a polynomial with a principal term. Then all the zeros of the function $H(z)$ defined in (6) have negative real parts if, and only if, all zeros of the function $F(y)$ defined by (7) are simple and real, and for each root y_0 of $F(y)$, $F'(y_0)G(y_0) < 0$.

Multiplying both sides of equation (2) by $e^{2\tau\lambda}$, then letting $\lambda = z/\tau$, we have

$$H(z) := z\left(e^{2z} + c\right) + Ae^{2z} + Ae^z = 0, \tag{8}$$

where $A = a\tau$. In equation (8), letting $z = iy$, and separating real and imaginary parts, we have

$$F(y) = -2\cos\frac{y}{2}\left[2y\sin\frac{y}{2}\cos y - A\cos\frac{y}{2}(2\cos y - 1)\right] = 0, \tag{9}$$

$$G(y) = y(\cos 2y + c) + A\sin 2y + A\sin y = 0. \tag{10}$$

By (9), we have

$$F'(y) = -2y\cos 2y - (1 + 2A)\sin 2y - A\sin y. \tag{11}$$

It is easy to see that in this case, the principal term of $F(y)$ is $y\psi_*^{(2)}(u,v) = y(2uv)$. Therefore, $\phi_*^{(2)}(z) = \sin 2z$.

LEMMA 2.3. $\phi_*^{(2)}(\pi/2 + iy) \neq 0$ for $y \in R$.

Indeed, $\phi_*^{(2)}(\pi/2 + iy) = -\cos 2iy = -\cosh 2y < 0$ for $y \in R$.

LEMMA 2.4. The solution of equation (1) satisfying the initial condition $x(\theta) = \phi(\theta)$ is given by

$$x(t) = X(t)[\phi(0) + c\phi(-2\tau)] - a\int_{-\tau}^{0} X(t - \tau - \theta)\phi(\theta)d\theta + c\int_{-2\tau}^{0} \phi(\theta)dX(t - \tau - \theta),$$

where the function $\phi(\theta)$ is a given continuously differentiable function defined on $[-2\tau,\ 0]$.

The function $X(t)$ is the solution of equation (1) with the initial data $X(\theta)=0,\ \theta<0,\ X(0)=1$, that is, the fundamental solution of equation (1), which can be expressed as

$$X(t)=\mathcal{L}^{-1}[h^{-1}(\lambda)]=\int_{(c)}\frac{e^{\lambda t}}{h(\lambda)}d\lambda,$$

where $h(\lambda)$ is defined in (2). The proof of Lemma 2.4 is similar to that of Theorem 7.5 of Chapter 1 in [3].

LEMMA 2.5. If $\alpha_0=\sup\{\mathrm{Re}\lambda:\ h(\lambda)=0\}$, then for any $\alpha>\alpha_0$, there is a constant $k=k(\alpha)$ such that the fundamental solution $X(t)$ of equation (1) satisfies the inequalities

$$|X(t)|\le ke^{\alpha t},\quad \mathrm{V}_{t-2\tau}^{t}X\le ke^{\alpha t},\quad t\ge 0.$$

where $\mathrm{V}_{t-2\tau}^{t}X$ denotes the total variation of $X(t)$ on $[t-2\tau,\ t]$.

The proof of Lemma 2.5 is similar to that of Theorem 7.6 of Chapter 1 in [3].

LEMMA 2.6. If $\alpha_0=\sup\{\mathrm{Re}\lambda:\ h(\lambda)=0\}$ and $x(\phi)(t)$ is the solution of equation (1) which coincides with ϕ on $[-2\tau\ 0]$, then for any $\alpha>\alpha_0$, there is a constant $k=k(\alpha)$ such that

$$|x(\phi)(t)|\le K|\phi|e^{\alpha t},\quad t\ge 0,\quad |\phi|=\sup_{-2\tau\le\theta\le 0}|\phi(\theta)|.$$

Lemma 2.6 follows immediately from Lemma 2.4 and Lemma 2.5. As a consequence, if $\alpha_0<0$, then all solutions of equation (1) decay exponentially.

Consider the functions

$$f_1(x):=-x+A\cot\frac{x}{2}\frac{2\cos x-1}{2\cos x},\tag{12}$$

$$f_2(x):=-x-A\tan\frac{x}{2}\frac{2\cos x+1}{2\cos x},\tag{13}$$

where $x\in(0,\pi/2)\cup(\pi/2,\pi),\ A>0$, and

$$g(y):=y^3-\frac{A+2}{2}y^2+\frac{A}{2}y-\frac{A}{2}=0.\tag{14}$$

PROPERTY 2.1. Let $A>0$, then the function $g(y)$ defined by (14) has only one real root in R. In particular, $g(y)<0$, for $y\in(-1,1)$.

Indeed, letting $y=z+(A+2)/6$, we see from $g(y)=0$ that

$$z^3+\left(-\frac{A^2}{12}+\frac{A}{6}-\frac{1}{3}\right)z+\left(-\frac{A^3}{108}+\frac{A^2}{36}-\frac{4A}{9}-\frac{2}{27}\right)=0.\tag{15}$$

The discriminant of equation (15) is

$$\begin{aligned}\Delta &= \left[\frac{1}{3}\left(-\frac{A^2}{12}+\frac{A}{6}-\frac{1}{3}\right)\right]^3+\left[\frac{1}{2}\left(-\frac{A^3}{108}+\frac{A^2}{36}-\frac{4A}{9}-\frac{2}{27}\right)\right]^2\\ &= \frac{A(3A^3-8A^2+80A+32)}{1728}.\end{aligned}$$

It is easy to verify that if $A > 0$, then $3A^3 - 8A^2 + 80A + 32 > 0$. Therefore, $\Delta > 0$. Consequently, equation (15), and hence $g(y)$ has only one real roots in R. On the other hand, by (14), we have $g(1) = -A/2 < 0$, for $A > 0$. Therefore, $g(y) < 0$ for $y < 1$. In particular, for $y \in (-1,\ 1)$, $g(y) < 0$. The proof of Property 2.1 is complete.

PROPERTY 2.2. Let $A > 0$, then the function $f_1(x)$, defined in (12), is decreasing on $(0,\ \pi/2)$ and $(\pi/2,\ \pi)$. Its range is $(-\infty, +\infty)$ when its domain is restricted to $(0, \pi/2)$ and is $(-\pi, +\infty)$ when restricted to $(\pi/2, \pi)$.

PROPERTY 2.3. Let $A > 0$, then the function $f_2(x)$ defined in (13) is decreasing on $(0,\ \pi/2)$ and $(\pi/2,\ \pi)$. Its range is $(-\infty,\ 0)$ when its domain is restricted to $(0,\ \pi/2)$ and is $(-\infty,\ +\infty)$ when restricted to $(\pi/2,\ \pi)$.

THEOREM 2.1. Let $A > 0$ and $0 \leq y < +\infty$. Then the function $f_1(x)$ defined in (12) has a double-valued inverse. Let its two single-valued branches be denoted respectively by $x = \theta_{01}(y; A)$ and $x = \theta_{02}(y; A)$. The function $\theta_{01}(y; A)$ is defined and decreasing in $[0, +\infty)$ with the range $(0, \theta_{01}(0; A)]$ and the function $\theta_{02}(y; A)$ is defined and decreasing in $[0, +\infty)$ with the range $(\pi/2, \theta_{02}(0; A)]$. They satisfy $0 < \theta_{01}(y; A) < \frac{\pi}{2} < \theta_{02}(y; A) < \pi$ and $\lim_{y\to+\infty} \theta_{01}(y; A) = 0$ and $\lim_{y\to+\infty} \theta_{02}(y;\ A) = \pi/2$.

Theorem 2.1 follows from Property 2.2 and the inverse function theorem.

THEOREM 2.2. Let $A > 0$ and $\pi \leq y < +\infty$. Then the function $f_2(x)$ defined in (13) has a unique inverse. Let it be denoted by $x = \theta_{11}(y; A)$. The function $\theta_{11}(y; A)$ is defined and decreasing in $[\pi, +\infty)$ with the range $(\pi/2, \theta_{11}(\pi; A)]$, satisfies $\pi/2 < \theta_{11}(y; A) < \pi$ and $\lim_{y\to+\infty} \theta_{11}(y; A) = \pi/2$.

Theorem 2.2 follows from Property 2.3 and the inverse function theorem.

3 Proofs of Main Theorems

We first find necessary and sufficient conditions for all the roots of equation (8) to have negative real parts.

LEMMA 3.1. A necessary condition for all the roots of equation (8) to have negative real parts is that $A > 0$.

Indeed, by (8), we have $H(0) = 2A$ and $\lim_{z\to+\infty} H(z) = +\infty$. Therefore, if $A \leq 0$, then equation (8) has at least one nonnegative real root. The proof is complete.

LEMMA 3.2. Equation (9) holds if, and only if, $\cos(y/2) = 0$ or

$$2m\pi = -x + A \cot \frac{x}{2} \frac{2\cos x - 1}{2\cos x}, \tag{16}$$

or

$$(2m + 1)\pi = -x - A \tan \frac{x}{2} \frac{2\cos x + 1}{2\cos x}, \tag{17}$$

where $x \in (0,\ \pi),\ m = 0, 1, 2, \ldots$.

PROOF. (9) holds if, and only if, $\cos(y/2) = 0$ or the equation

$$y = A \cot \frac{y}{2} \frac{2\cos y - 1}{2\cos y} \tag{18}$$

holds. In (18), letting

$$y = 2m\pi + x, \tag{19}$$

where $x \in (0,\ \pi),\ m = 0, 1, ...,$ we have (16), and letting

$$y = (2m+1)\pi + x, \tag{20}$$

where $x \in (0,\ \pi),\ m = 0, 1, ...,$ we have (17) respectively. The proof is complete.

LEMMA 3.3. Let $A > 0$. (i) All the real roots of equation (16) are given by $x_{2m}^{(01)} = \theta_{01}(2m\pi; A)$ and $x_{2m}^{(02)} = \theta_{02}(2m\pi; A)$, $m = 0, 1, 2, ...$, where both $\{\theta_{01}(2m\pi; A)\}$ and $\{\theta_{02}(2m\pi; A)\}$ are decreasing sequences, which satisfy $0 < \theta_{01}(2m\pi; A) < \frac{\pi}{2} < \theta_{02}(2m\pi; A) < \pi$, and $\lim_{m\to+\infty} \theta_{01}(2m\pi; A) = 0$, $\lim_{m\to+\infty} \theta_{02}(2m\pi; A) = \pi/2$. (ii) All the real roots of equation (17) are given by $x_{2m+1}^{(11)} = \theta_{11}((2m+1)\pi; A)$, $m = 0, 1, 2, ...$, where $\{\theta_{11}((2m+1)\pi; A)\}$ is a decreasing sequence which satisfies $\frac{\pi}{2} < \theta_{11}((2m+1)\pi; A) < \pi$, and $\lim_{m\to+\infty} \theta_{11}((2m+1)\pi; A) = \pi/2$.

Indeed, in (12), taking $f_1(x) = 2m\pi$, we have equation (16). Consequently, by Theorem 2.1, we have Part (i). In (13), taking $f_2(x) = (2m+1)\pi$, we have equation (17). Consequently, by Theorem 2.2, we have Part (ii).

LEMMA 3.4. Let $A > 0$. Then all the real roots of equation (9) are given by

$$\begin{aligned} y^{(\pm)}(2m+1) &= \pm(2m+1)\pi, \\ y_{01}^{(\pm)}(2m) &= \pm[2m\pi + \theta_{01}(2m\pi; A)], \\ y_{02}^{(\pm)}(2m) &= \pm[2m\pi + \theta_{02}(2m\pi; A)], \\ y_{11}^{(\pm)}(2m+1) &= \pm[(2m+1)\pi + \theta_{11}((2m+1)\pi; A)], \end{aligned} \tag{21}$$

$m = 0, 1, 2, ...$, where the sequences $\{\theta_{01}(2m\pi; A)\}$ and $\{\theta_{02}(2m\pi; A)\}$ are defined in Lemma 3.3(i) and the sequence $\{\theta_{11}((2m+1)\pi; A)\}$ is defined in Lemma 3.3(ii).

Indeed, it is clear that $\pm(2m+1)\pi,\ m = 0, 1, 2, ...$, are all the real roots of equation (12). Consequently, by Lemma 3.3, (19), (20) and Lemma 3.2, we know that (21) gives all the real roots of equation (9).

LEMMA 3.5. Let $A > 0$. Then equation (9) has $8k + 1$ real roots in the strip $-2k\pi + \pi/2 \le \mathrm{Re}z < 2k\pi + \pi/2$, $k = 1, 2, ...$.

Lemma 3.5 follows immediately from Lemma 3.4 .

Set

$$E(y) = G(y)F'(y), \tag{22}$$

where $G(y)$ and $F'(y)$ are defined by (10) and (11). Substituting all the real roots of equation (9) given by Lemma 3.4 into (22), we have

$$E(y^{(\pm)}(2m+1)) = -(2m+1)^2\pi^2(1+c),\ m = 0, 1, ... \tag{23}$$

$$\begin{aligned} E(y_{0j}^{(\pm)}(2m)) = &[(2m\pi + \theta_{0j})(\cos 2\theta_{0j} + c) + A\sin 2\theta_{0j} + A\sin\theta_{0j}] \\ &[-2(2m\pi + \theta_{0j})\cos 2\theta_{0j} - (1+2A)\sin 2\theta_{0j} - A\sin\theta_{0j}], \end{aligned} \tag{24}$$

where $\theta_{0j} := \theta_{0j}(2m\pi; A),\ j = 1, 2;\ m = 0, 1, ...,$ and

$$\begin{aligned} & E(y_{11}^{(\pm)}(2m+1)) \\ = \ & [((2m+1)\pi + \theta_{11})(\cos 2\theta_{11} + c) + A \sin 2\theta_{11} - A \sin \theta_{11}] \\ & [-2((2m+1)\pi + \theta_{11}) \cos 2\theta_{11} - (1 + 2A) \sin 2\theta_{11} + A \sin \theta_{11}], \end{aligned} \tag{25}$$

where $\theta_{11} := \theta_{11}((2m+1)\pi; A),\ m = 0, 1, 2, ...$. By Lemma 3.3(i), we have

$$2m\pi \equiv -\theta_{0j} + A \cot \frac{\theta_{0j}}{2} \frac{2\cos\theta_{0j} - 1}{2\cos\theta_{0j}} \tag{26}$$

where $\theta_{0j} := \theta_{0j}(2m\pi; A),\ j = 1, 2;\ m = 0, 1, 2, ...$. By Lemma 3.3(ii), we have

$$(2m+1)\pi \equiv -\theta_{11} - A \tan \frac{\theta_{11}}{2} \frac{2\cos\theta_{11} + 1}{2\cos\theta_{11}}, \tag{27}$$

where $\theta_{11} := \theta_{11}((2m+1)\pi; A),\ m = 0, 1, 2, ...$. Substituting (26) into (24), we have

$$\begin{aligned} & E(y_{0j}^{(\pm)}(2m)) \\ = \ & \left[A \cot \frac{\theta_{0j}}{2} \frac{2\cos\theta_{0j} - 1}{2\cos\theta_{0j}} (\cos 2\theta_{0j} + c) + A \sin 2\theta_{0j} + A \sin \theta_{0j}\right] \\ & \left[-2A \cot \frac{\theta_{0j}}{2} \frac{2\cos\theta_{0j} - 1}{2\cos\theta_{0j}} \cos 2\theta_{0j} - (1 + 2A) \sin 2\theta_{0j} - A \sin \theta_{0j}\right] \\ = \ & \frac{2A(1 + \cos\theta_{0j})^2 (2c \cos\theta_{0j} + 1 - c)}{\sin^2 2\theta_{0j}} \\ & (2\cos^3\theta_{0j} - (A+2)\cos^2\theta_{0j} + A\cos\theta_{0j} - A) \end{aligned} \tag{28}$$

where $\theta_{0j} := \theta_{0j}(2m\pi; A),\ j = 1, 2;\ m = 0, 1, 2, ...$. Substituting (27) into (25), we have

$$\begin{aligned} & E(y_{11}^{(\pm)}(2m+1)) \\ = \ & \left[-A \tan \frac{\theta_{11}}{2} \frac{2\cos\theta_{11} + 1}{2\cos\theta_{11}} (\cos 2\theta_{11} + c) + A \sin 2\theta_{11} - A \sin \theta_{11}\right] \\ & \left[2A \tan \frac{\theta_{11}}{2} \frac{2\cos\theta_{11} + 1}{2\cos\theta_{11}} \cos 2\theta_{11} - (1 + 2A) \sin 2\theta_{11} + A \sin \theta_{11}\right] \\ = \ & \frac{2A(1 - \cos\theta_{11})^2 (2c \cos\theta_{11} - 1 + c)}{\sin^2 2\theta_{11}} \\ & (2\cos^3\theta_{11} + (A+2)\cos^2\theta_{11} + A\cos\theta_{11} + A) \end{aligned} \tag{29}$$

where $\theta_{11} := \theta_{11}((2m+1)\pi; A),\ m = 0, 1, 2, ...$.

THEOREM 3.1. All roots of equation (8) have negative real parts if, and only if, its parameters satisfy $A > 0,\ 0 < |c| < 1$, and, either $-1 < c \le 1/3$ and $c \ne 0$, or, the following conditions

$$\frac{1}{3} < c < 1,\ -\arccos \frac{1-c}{2c} + \frac{A}{1-c} \sqrt{\frac{3c-1}{1+c}} < 0 \tag{30}$$

hold.

PROOF. In view of Lemma 3.1, Lemma 3.5 and Lemma 2.1, when $A > 0$, all the roots of equation (9) are simple and real. Therefore, by Lemma 2.3 and Lemma 2.2, it suffices to find equivalent conditions for the right hand sides of (23), (28) and (29) to be less than zero, that is,

$$-(2m+1)^2\pi^2(1+c) < 0,\ m = 0, 1, ..., \tag{31}$$

$$\frac{2A(1+\cos\theta_{0j})^2}{\sin^2 2\theta_{0j}}(2c\cos\theta_{0j}+1-c)(2\cos^3\theta_{0j}-(A+2)\cos^2\theta_{0j}+A\cos\theta_{0j}-A) < 0, \tag{32}$$

where $\theta_{0j} := \theta_{0j}(2m\pi; A),\ j = 1, 2;\ m = 0, 1, 2, ...,$ and

$$\frac{2A(1-\cos\theta_{11})^2}{\sin^2 2\theta_{11}}(2c\cos\theta_{11}-1+c)(2\cos^3\theta_{11}+(A+2)\cos^2\theta_{11}+A\cos\theta_{11}+A) < 0, \tag{33}$$

where $\theta_{11} := \theta_{11}((2m+1)\pi; A),\ m = 0, 1, 2, ...$. It is easy to see that (31) is equivalent to $c > -1$. Next, by Lemma 3.3(i), we know that

$$0 < \theta_{01}(2m\pi; A) < \frac{\pi}{2} < \theta_{02}(2m\pi; A) < \pi,\ m = 0, 1, ... \tag{34}$$

By (34) and Property 2.1, we have $2\cos^3\theta_{0j} - (A+2)\cos^2\theta_{0j} + A\cos\theta_{0j} - A < 0$, where $\theta_{0j} := \theta_{0j}(2m\pi; A),\ j = 1, 2;\ m = 0, 1, 2, ...$. It follows that (32) is equivalent to

$$2c\cos\theta_{0j}(2m\pi; A) + 1 - c > 0,\ j = 1, 2;\ m = 0, 1, 2, ... \tag{35}$$

There are two cases: $-1 < c < 0$ and $c > 0$. In the former case, by Lemma 3.3(i), we know that $\theta_{0j}(2m\pi; A)$, $j = 1, 2$, are decreasing sequences, $\lim_{m\to+\infty}\theta_{01}(2m\pi; A) = 0$, $\lim_{m\to+\infty}\theta_{02}(2m\pi; A) = \pi/2$, and (34) holds. Therefore, the left-hand side of (35) are also decreasing sequences, consequently, when $j = 1$, (35) is equivalent to

$$\lim_{m\to+\infty}(2c\cos\theta_{01}(2m\pi; A) + 1 - c) = 2c + 1 - c = c + 1 \geq 0, \tag{36}$$

and when $j = 2$, (35) is equivalent to

$$\lim_{m\to+\infty}(2c\cos\theta_{02}(2m\pi; A) + 1 - c) = 1 - c \geq 0. \tag{37}$$

In the latter case $c > 0$, the left-hand side of (35) are increasing sequences, and consequently, are equivalent to

$$\cos\theta_{0j}(0; A) > (c-1)/(2c),\ j = 1, 2. \tag{38}$$

Since $0 < c \leq 1/3$ if, and only if, $(c-1)/(2c) \leq -1$, therefore, under the condition $0 < c \leq 1/3$, (38) holds. On the other hand, $1/3 < c < 1$ is equivalent to $-1 < (c-1)/(2c) < 0$.Therefore, under the condition $1/3 < c < 1$, by (34), we have

$$\cos\theta_{01}(0; A) > \frac{c-1}{2c}. \tag{39}$$

The inequality

$$\cos\theta_{02}(0;A) > \frac{c-1}{2c} \tag{40}$$

is equivalent to

$$\theta_{02}(0;A) < \arccos\frac{c-1}{2c},$$

that is,

$$0 > -\arccos\frac{c-1}{2c} + A\cot\frac{\arccos\frac{c-1}{2c}}{2}\,\frac{2\cos\arccos\frac{c-1}{2c}-1}{2\cos\arccos\frac{c-1}{2c}},$$

that is,

$$0 > -\arccos\frac{c-1}{2c} + \frac{A}{1-c}\sqrt{\frac{3c-1}{1+c}}. \tag{41}$$

Since $1 \le c$ is equivalent to $0 \le (c-1)/(2c) < 1$, therefore, by (34), we know that under the condition $c \ge 1$, (40) does not hold. Summarizing the above discussion, we see that (35) is equivalent to either (3) or (30) when $A > 0$.

Next, we will find the equivalent conditions for (33) to hold. By Lemma 3.3(ii), we know that

$$\frac{\pi}{2} < \theta_{11}((2m+1)\pi;A) < \pi. \tag{42}$$

Set

$$g_1(y) = 2y^2 + (A+2)y^2 + Ay + A,$$

then

$$g_1(y) = -2g(-y), \tag{43}$$

where $g(y)$ has been defined by (14). From (42), (43) and Property 3.1, it follows that

$$2\cos^3\theta_{11} + (A+2)\cos^2\theta_{11} + A\cos\theta_{11} + A > 0. \tag{44}$$

where $\theta_{11} := \theta_{11}((2m+1)\pi;A),\ m = 0,1,\dots$. By (42) and (44), we know that (33) is equivalent to

$$2c\cos\theta_{11}((2m+1)\pi;A) - 1 + c < 0,\ m = 0,1,\dots\,. \tag{45}$$

By Lemma 3.3(ii), we know that $\theta_{11}((2m+1)\pi;A)$ is a decreasing sequence, and $\lim_{m\to+\infty}\theta_{11}((2m+1)\pi;A) = \pi/2$, and (42) holds. Therefore, if $-1 < c < 0$, then the left-hand side of (45) is also a decreasing sequence. Consequently, (45) is equivalent to

$$\cos\theta_{11}(\pi;A) > \frac{1-c}{2c}. \tag{46}$$

On the other hand, it is easy to see that $-1 < c < 0$ is equivalent to $(1-c)/(2c) < -1$. Therefore, under the condition $-1 < c < 0$, (46) always holds. If $c > 0$, then the left-hand side of (45) is an increasing sequence. Therefore, (45) is equivalent to

$$\lim_{m\to+\infty}(2c\cos\theta_{11}((2m+1)\pi;A) - 1 + c) = -1 + c \le 0. \tag{47}$$

Combining (46) and (47), we see that (45) is equivalent to $-1 < c \le 1$ and $c \ne 0$ when $A > 0$. The proof is complete.

COROLLARY 3.1. All the roots of equation (2) have negative real parts if, and only if, its parameters satisfy $a > 0,\ 0 < |c| < 1$, and either (3) or (4) holds.

PROOF OF THEOREM 1.1. Set $\Lambda = \{\mathrm{Re}\,\lambda : h(\lambda) = 0\}$ and $\alpha_0 = \sup\Lambda$, where $h(\lambda)$ is defined in (2). The comparison equation of (2) is

$$\lambda(1 + ce^{-2\tau\lambda}) = 0.$$

It is easy to see that all its roots are given by $\lambda = 0$ and $\lambda = (\ln|c| \pm k\pi i)/(2\tau)$, $k = 0, 1, 2, \ldots$. Therefore, $(\ln|c|)/(2\tau)$ is the unique limit point of the set Λ. It is well known that the set Λ is bounded. If the set Λ can reach its supremum, that is, $\sup\Lambda = \max\Lambda$, then the statement that all the roots of equation (2) have negative real parts is equivalent to $\alpha_0 = \sup\Lambda < 0$. On the other hand, if the set Λ cannot reach its supremum, then $\alpha_0 = \sup\Lambda = (\ln|c|)/(2\tau)$. By Corollary 3.1, we know that the statement that all the roots of equation (2) have negative real parts implies that $|c| < 1$. Therefore, in this case, we also obtain the conclusion that $\alpha_0 = \sup\Lambda < 0$. From the above discussion and Lemma 2.6, it follows that if all the roots of equation (2) have negative real parts, then the trivial solution of equation (1) is exponentially asymptotic stable.

The converse is easily seen. The proof of Theorem 1.1 is complete.

Finally, Theorem 1.2 follows immediately From Corollary 3.1 and Theorem 1.1.

References

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