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On the Rate of Pointwise Summability of Fourier Series *

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Abstract

We introduce a local modulus of continuity as a measure of pointwise summability by Abel and (C,2) means. The (C,1) summability is also considered.

1 Introduction

Let L^p (C) be the class of all 2π–periodic real functions integrable in the Lebesgue sense with p–th power (respectively continuous functions) over $Q = [-\pi, \pi]$. Let $X = X^p$ where $X^p = L^p$ when $1 \leq p < \infty$ or $X^p = C$ when $p = \infty$. Let us define the norm of $f \in X^p$ as

$$\|f\|_{X^p} := \|f(\cdot)\|_{X^p} = \begin{cases} \left(\int_Q \mid f(x) \mid^p dx\right)^{1/p} & 1 \leq p < \infty \\ \sup_{x \in Q} \mid f(x) \mid & p = \infty \end{cases} .$$

Let

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} (a_k(f) \cos kx + b_k(f) \sin kx) := \sum_{k=0}^{\infty} C_k f(x)$$

and let $S_k f$, $\sigma_k^\alpha f$ and $A_r f$ be the partial sum, the (C, α) mean and the Abel mean of the trigonometric Fourier series Sf respectively. Thus

$$\begin{aligned} \sigma_n^0 f &= S_n f,\ n = 0, 1, 2, ..., \\ \sigma_n^\alpha f &= \frac{1}{A_n^\alpha} \sum_{k=0}^{n} A_{n-k}^\alpha C_k f,\ \alpha > -1, n = 0, 1, 2, ..., \end{aligned}$$

$$A_k^\alpha = C_{k+\alpha}^k = \begin{pmatrix} k + \alpha \\ k \end{pmatrix}$$

and

$$A_r f = \sum_{k=0}^{\infty} r^k C_k f,\ r \in (0, 1) .$$

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The pointwise characteristic

$$\overline{w}_x f_p(\delta) = \sup_{0<h\leq\delta} \left\{ \frac{1}{h} \int_0^h |\varphi_x(t)|^p \, dt \right\}^{1/p},$$

where

$$\varphi_x(t) = f(x+t) + f(x-t) - 2f(x),$$

was used as a measure of approximation by Aljančič et al. [1]. This characteristic was very often used, but it appears that such approximation cannot be comparable with the norm approximation when $X \neq C$. In [5] we introduced the modified quantity

$$w_x f_p(\delta) := \left\{ \frac{1}{\delta} \int_0^\delta |\varphi_x(t)|^p \, dt \right\}^{1/p}.$$

In view of the monotonicity of the product $\delta w_x f_1(\delta)$ with respect to δ, by the methods in [5], we may easily obtain the following estimation

$$\left|\sigma_n^1 f(x) - f(x)\right| \leq \frac{3/2}{n+1} \sum_{k=0}^{n} w_x f_1\left(\frac{\pi}{k+1}\right), \; n = 0, 1, 2, \ldots .$$

Based on the points of differentiability of the indefinite integral of f (D-points), the quantity

$$\overline{w}_x^* f(\delta) := \sup_{0<h\leq\delta} \left| \frac{1}{h} \int_0^h \varphi_x(t) \, dt \right|$$

is considered in [7].

Here we introduce yet another modified quantity

$$w_x^* f(\delta) := \sup_{0<h\leq\delta} \left| \frac{1}{\delta} \int_0^h \varphi_x(t) \, dt \right|.$$

We can observe that for $p \in [1, \infty)$,

$$w_x f_p(\delta) \leq \overline{w}_x f_p(\delta) \leq \omega f_C(\delta),$$

and

$$w_x^* f(\delta) \leq \overline{w}_x^* f(\delta) \leq \omega f_C(\delta),$$

and also, for $\widetilde{p} \in [p, \infty]$, by the Minkowski inequality,

$$\|w_\cdot^* f(\delta)\|_{X^{\widetilde{p}}} \leq \|w_\cdot f_p(\delta)\|_{X^{\widetilde{p}}} \leq \omega f_{X^{\widetilde{p}}}(\delta),$$

where ωf_X is the modulus of continuity of f in the space $X = X^{\widetilde{p}}$ defined by the formula

$$\omega f_X(\delta) := \sup_{0<|h|\leq\delta} \|f(\cdot+h) - f(\cdot)\|_X .$$

It is well-known (see [2,3,4]) that the $(C,1)$ means do not tend to f at the D-points of f but the Abel means as well the $(C,2)$ means do. In this paper these facts will be presented in the approximation version with the quantity $w_x^* f$ as a measure of such approximation (cf. [6]).

By K we shall designate either an absolute constant or a constant depending on the indicated parameters, not necessarily the same in each occurrence.

2 Results

We start with two theorems for pointwise approximation.

THEOREM 1. If $f \in L^1$, then

$$\left|\sigma_n^2 f(x) - f(x)\right| \leq K \frac{1}{n+1} \sum_{k=0}^{n} w_x^* f\left(\frac{\pi}{k+1}\right),$$

for all real x and every positive integer n.

THEOREM 2. If $f \in L^1$, then

$$\left|A_r f(x) - f(x)\right| \leq K \left(w_x^* f(\pi)(1-r) + (1-r) \int_{1-r}^{\pi} \frac{w_x^* f(t)}{t^2} dt \right)$$

for all real x and every $r \in (0,1)$.

Now using the Minkowski inequality, we can derive from these theorems a corollary.

COROLLARY 1. If $f \in X = X^p$ $(p \in [1,\infty])$, then

$$\left\|\sigma_n^2 f - f\right\|_X \leq K \frac{1}{n+1} \sum_{k=0}^{n} \omega f_X \left(\frac{\pi}{k+1}\right)$$

and

$$\left\|A_r f - f\right\|_X \leq K \left((1-r)\, \omega f_X(\pi) + (1-r) \int_{1-r}^{\pi} \frac{\omega f_X(t)}{t^2} dt \right)$$

for every integer n and $r \in (0,1)$ respectively.

From our theorems the results of Fatou [3] and Lebesgue [4] also follow.

COROLLARY 2. If $f \in L$ and x is a D-point of f, then $\sigma_n^2 f(x) - f(x) = o_x(1)$ as $n \to \infty$ and $A_r f(x) - f(x) = o_x(1)$ as $r \to 1^-$.

To prove Theorem 1, let us observe that

$$\sigma_n^2 f(x) - f(x) = \frac{1}{A_n^2} \sum_{\nu=0}^{n} A_\nu^1 \left(\sigma_\nu^1 f(x) - f(x)\right) = \frac{1}{2\pi A_n^2} \int_0^{\pi} \varphi_x(t) \sum_{\nu=0}^{n} \left(\frac{\sin \frac{\nu+1}{2} t}{\sin \frac{1}{2} t} \right)^2 dt.$$

Putting

$$\begin{aligned} G_n(0) &= \sum_{\nu=0}^{n} (\nu+1)^2 \\ G_n(t) &= \sum_{\nu=0}^{n} \left(\frac{\sin \frac{\nu+1}{2} t}{\sin \frac{1}{2} t} \right)^2, \; t \neq 0, \end{aligned}$$

and integrating by parts we obtain

$$\sigma_n^2 f(x) - f(x) = \frac{1}{2\pi A_n^2}\left\{\left[\int_0^t \varphi_x(u)\,du G_n(t)\right]_0^\pi - \int_0^\pi \left(\int_0^t \varphi_x(u)\,du\right)\frac{d}{dt}G_n(t)\,dt\right\}.$$

An easy computation yields

$$\sum_{\nu=0}^{n} \sin^2 \frac{\nu+1}{2}t = \frac{2n+3}{4} - \frac{\sin\frac{2n+3}{2}t}{4\sin\frac{1}{2}t},$$

and therefore

$$G_n(t) = \frac{2n+3}{4\sin^2\frac{1}{2}t} - \frac{\sin\frac{2n+3}{2}t}{4\sin^3\frac{1}{2}t},$$

whence

$$G_n(\pi) = \frac{2n+3}{4} - \frac{(-1)^{n+1}}{4}.$$

Thus

$$\begin{aligned} &\left|\sigma_n^2 f(x) - f(x)\right| \\ \leq\ & \frac{2n+3+(-1)^n}{4(n+1)^2} w_x^* f(\pi) + \frac{1}{\pi(n+1)^2}\left(\int_0^{\frac{\pi}{n+1}} + \int_{\frac{\pi}{n+1}}^{\pi}\right) t w_x^* f(t)\left|\frac{d}{dt}G_n(t)\right| dt. \end{aligned}$$

Let S_1, S_2 and S_3 denote respectively the first, second and the third sum in the right hand side of the above inequality. Immediately we have

$$S_1 \leq \frac{1}{n+1} w_x^* f(\pi) \leq \frac{1}{n+1}\sum_{k=0}^{n} w_x^* f\left(\frac{\pi}{k+1}\right).$$

Differentiating G_n we obtain

$$\frac{d}{dt}G_n(t) = \frac{-(2n+3)\cos\frac{1}{2}t - \frac{2n+3}{2}\cos\frac{2n+3}{2}t}{4\sin^3\frac{1}{2}t} + \frac{3\cos\frac{1}{2}t\sin\frac{2n+3}{2}t}{8\sin^4\frac{1}{2}t},$$

whence

$$\left|\frac{d}{dt}G_n(t)\right| \leq \frac{4.5(2n+3)}{4\sin^3\frac{1}{2}t} \leq \frac{9\pi^2}{4}\frac{n+1}{t^3},\ 0 < |t| \leq \pi,$$

and

$$\frac{d}{dt}G_n(t) = \frac{d}{dt}\left(2\sum_{\nu=0}^{n}\sum_{k=0}^{\nu}\left(\frac{1}{2} + \sum_{\mu=1}^{k}\cos\mu t\right)\right) = -2\sum_{\nu=0}^{n}\sum_{k=0}^{\nu}\sum_{\mu=1}^{k}\mu\sin\mu t,$$

Thus, for all t,

$$\left|\frac{d}{dt}G_n\left(t\right)\right| \leq 2\sum_{\nu=0}^{n}\sum_{k=0}^{\nu}\sum_{\mu=1}^{k}\mu^2\sin t \leq 2n^5\sin t \leq 2\left(n+1\right)^5 t.$$

These imply

$$S_2 \leq \frac{1}{\pi\left(n+1\right)^2}\frac{\pi}{n+1}w_x^*f\left(\frac{\pi}{n+1}\right)2\left(n+1\right)^5\int_0^{\frac{\pi}{n+1}} tdt \leq 2\pi^2 w_x^*f\left(\frac{\pi}{n+1}\right),$$

and

$$S_3 \leq \frac{9\pi^2}{4\pi\left(n+1\right)^2}\int_{\frac{\pi}{n+1}}^{\pi} tw_x^*f(t)\frac{n+1}{t^3}dt = \frac{9\pi}{4}\frac{1}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{w_x^*f(t)}{t^2}dt.$$

Finally, by the monotonicity of $tw_x^*f(t)$,

$$w_x^*f(\frac{\pi}{n+1}) = \frac{\pi}{n+1}w_x^*f(\frac{\pi}{n+1})\frac{4\pi}{n+1}\int_{\frac{\pi}{n+1}}^{\pi} t^{-3}dt \leq \frac{4\pi}{n+1}\int_{\frac{\pi}{n+1}}^{\pi}\frac{w_x^*f(t)}{t^2}dt$$

and

$$\begin{aligned}
\int_{\frac{\pi}{n+1}}^{\pi}\frac{w_x^*f(t)}{t^2}dt &= \frac{1}{\pi}\int_1^{n+1} w_x^*f(\frac{\pi}{u})du = \frac{1}{\pi}\sum_{k=1}^{n}\int_k^{k+1} u\sup_{0<h\leq\pi/u}\left|\int_0^h \varphi_x\left(t\right)dt\right|du \\
&\leq \frac{1}{\pi}\sum_{k=1}^{n}\left(k+1\right)\sup_{0<h\leq\pi/k}\left|\int_0^h \varphi_x\left(t\right)dt\right| \\
&\leq 2\sum_{k=1}^{n}w_x^*f(\frac{\pi}{k}) \leq 2\sum_{k=0}^{n}w_x^*f(\frac{\pi}{k+1}).
\end{aligned}$$

These estimates imply our result with constant $K = 1 + 16\pi^3 + 9\pi/2$. The proof is complete.

Next, we prove Theorem 2. First of all, it is clear that

$$A_r f\left(x\right) - f\left(x\right) = \frac{1}{\pi}\int_0^{\pi}\varphi_x\left(t\right)P\left(r,t\right)dt,$$

where

$$P\left(r,t\right) = \frac{1-r^2}{2\Delta\left(r,t\right)}$$

and

$$\Delta\left(r,t\right) = 1 - 2r\cos t + r^2 = \left(1-r\right)^2 + 4r\sin^2\frac{1}{2}t.$$

Integration by parts gives

$$\begin{aligned}
& A_r f\left(x\right) - f\left(x\right) \\
= & \left[\frac{1}{\pi}\int_0^t \varphi_x\left(u\right)duP\left(r,t\right)\right]_{t=0}^{\pi} - \frac{1}{\pi}\int_0^{\pi}\left(\int_0^t \varphi_x\left(u\right)du\right)\frac{\partial}{\partial t}P\left(r,t\right)dt \\
= & \frac{1-r}{2\left(1+r\right)}\frac{1}{\pi}\int_0^{\pi}\varphi_x\left(u\right)du + \frac{1}{\pi}\int_0^{\pi}\left(\int_0^t \varphi_x\left(u\right)du\right)\frac{r\left(1-r^2\right)\sin t}{\Delta^2\left(r,t\right)}dt.
\end{aligned}$$

Hence,

$$\begin{aligned}\left|A_r f(x) - f(x)\right| &\leq (1-r)\, w_x^* f(\pi) + \frac{1}{\pi}\int_0^\pi t w_x^* f(t) \frac{r\left(1-r^2\right)\sin t}{\Delta^2(r,t)} dt \\ &= (1-r)\, w_x^* f(\pi) + \frac{1}{\pi}\left(\int_0^{1-r} + \int_{1-r}^{\pi}\right) t w_x^* f(t) \frac{r\left(1-r^2\right)\sin t}{\Delta^2(r,t)} dt \\ &= (1-r)\, w_x^* f(\pi) + I_1 + I_2.\end{aligned}$$

Using the estimate $\Delta(r,t) \geq (1-r)^2$ we can see that

$$I_1 \leq \frac{1}{\pi}(1-r)\, w_x^* f(1-r) \frac{r\left(1-r^2\right)(1-r)}{(1-r)^4} \int_0^{1-r} dt \leq \frac{2}{\pi} w_x^* f(1-r).$$

By the estimate $\Delta(r,t) \geq \frac{4r}{\pi^2} t^2$ we obtain

$$I_2 \leq \frac{1}{\pi}\int_{1-r}^{\pi} t w_x^* f(t) \frac{r\left(1-r^2\right)t}{\frac{16r^2}{\pi^4}t^4} dt = \frac{\pi^3}{8r}(1-r)\int_{1-r}^{\pi} \frac{w_x^* f(t)}{t^2} dt.$$

Finally, because $t w_x^* f_1(t)$ is a non-decreasing function of t,

$$w_x^* f(1-r) \leq \frac{\pi^2-1}{2\pi^2} w_x^* f(1-r)(1-r)^2 \int_{1-r}^{\pi} t^{-3} dt \leq \frac{\pi^2-1}{2\pi^2}(1-r)\int_{1-r}^{\pi} \frac{w_x^* f(t)}{t^2} dt,$$

which completes our proof.

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