Applied Mathematics E-Notes, 1(2001), 34-39 ©
Available free at mirror sites of http://math2.math.nthu.edu.tw/∼amen/

Two Special Convolution Products of $(n/2-k-1)$-th Derivatives of Dirac Delta in Hypercone *

Manuel A. Aguirre T.†

Received 25 October 2000

Abstract

In this paper two special convolution products $\delta^{(n/2-k-1)}(u)*\delta^{(n/2-l-1)}(u)$ and $\delta^{(n/2-k-1)}(m^2+u)*\delta^{(n/2-l-1)}(m^2+u)$ are expressed in terms of several known quantities.

Let $x=(x_1,x_2,...x_n)$ be a point of R^n. We shall write

$$x_1^2+...+x_\mu^2-x_{\mu+1}^2-...x_{\mu+\nu}^2=u, \tag{1}$$

where $\mu+\nu=n$. Γ_+ denotes the interior of the forward cone

$$\Gamma_+=\{x\in R^n|\ x_1>0,u>0\}, \tag{2}$$

and $\bar{\Gamma}_+$ denotes its closure. Similarly, Γ_- denotes the domain

$$\Gamma_-=\{x\in R^n|\ x_1<0,u>0\} \tag{3}$$

and $\bar{\Gamma}_-$ denotes its closure. Let $F(\lambda)$ be a function of the scalar variable λ, and let $\Phi=\Phi(x)$ be a function endowed with the following properties: (i) $\Phi(x)=F(u)$, (ii) supp$\Phi(x)\subset\bar{\Gamma}_+$ and (iii) $e^{\langle x,y\rangle}\Phi(x)\in L_1$ if $y\in V_-$, where

$$V_-=\left\{y\in R^n|\ y_1>0,y_1^2+...+y_\mu^2-y_{\mu+1}^2-...y_{\mu+\nu}^2>0\right\}. \tag{4}$$

We let R denote the family of functions Φ.

Similarly, A denotes the family of functions of the form $\Phi=\Phi(x)$ which satisfies conditions: (a) $\Phi(x)=F(u)$, (b) supp$\Phi(x)\subset\bar{\Gamma}_-$, and (c) $e^{\langle x,y\rangle}\Phi(x)\in L_1$ if $y\in V_+$, where

$$V_+=\left\{y\in R^n|\ y_1<0,y_1^2+...+y_\mu^2-y_{\mu+1}^2-...y_{\mu+\nu}^2>0\right\}. \tag{5}$$

We shall consider the following functions of the family R_α introduced in [1, p.72]: $R_\alpha(u)=0$ if $x\notin\Gamma_+$, and

$$R_\alpha(u)=\frac{1}{K_n(\alpha)}u^{(\alpha-n)/2},\ x\in\Gamma_+.$$

*Mathematics Subject Classifications: 46F10, 46F12.

†(Núcleo Consolidado de Matemática Pura y Aplicada) Facultad de Ciencias Exactas, Universidad Nacional del Centro Pinto, 399, (7000) Tandil, Argentina, email:maguirre@exa.unicen.edu.ar

Here α is a complex parameter and n the dimension of the space, the constant $K_n(\alpha)$ is defined by

$$K_n(\alpha) = \frac{\pi^{(n-1)/2}\Gamma((2+\alpha-n)/2)\Gamma((1-\alpha)/2))\Gamma(\alpha)}{\Gamma((2+\alpha-\mu)/2)\Gamma((\mu-\alpha)/2)} \tag{6}$$

and μ is the number of positive terms of

$$u = x_1^2 + ... + x_\mu^2 - x_{\mu+1}^2 - ... - x_{\mu+\nu}^2,\ \mu+\nu = n. \tag{7}$$

$R_\alpha(u)$ is a distribution of α and is an ordinary function if the real part of α is greater than or equal to n.

By putting $\mu = 1$ in $R_\alpha(u)$ and (6) and remembering the Legendre's duplication formula of $\Gamma(z)$ [2, p.344]

$$\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma(z+1/2), \tag{8}$$

$R_\alpha(u)$ reduces to $M_\alpha(u)$ which is the hyperbolic kernel of Riesz [3, p.31]: $M_\alpha(u) = 0$ if $x \notin \Gamma_+$ and

$$M_\alpha(u) = \frac{1}{H_n(\alpha)} u^{(\alpha-n)/2},\ x \in \Gamma_+. \tag{9}$$

Here

$$u = x_1^2 - x_2^2 - ...x_n^2, \tag{10}$$

and

$$H_n(\alpha) = \pi^{(n-2)/2}2^{\alpha-1}\Gamma(\alpha/2)\Gamma((\alpha-n+2)/2). \tag{11}$$

Trione in [4, p.11] proves the validity of the property

$$\diamondsuit^k R_{2k}(u) = R_0(u) = \delta(x) \tag{12}$$

for $k = 0, 1, 2, ...$, where

$$\diamondsuit^k = \left\{ \frac{\partial^2}{\partial x_1^2} + ... \frac{\partial^2}{\partial x_\mu^2} - \frac{\partial^2}{\partial x_{\mu+1}^2} - ... \frac{\partial^2}{\partial x_{\mu+\nu}^2} \right\}^k \tag{13}$$

is the ultrahyperbolic operator iterated k-times and $\delta(x) = \delta(x_1, x_2, ...x_n)$ is the Dirac delta function. From (12), $R_{2k}(u)$ is the unique elementary solution of the n-dimensional ultrahyperbolic operator iterated k-times defined by (13)

Aguirre in [5, p.149] proves the following properties:

1. μ odd and ν even (n odd)

$$R_{2k}(u) = \frac{1}{(-1)^{(\mu-1)/2}\pi^{(n-1)/2}2^{2k-1}\Gamma(k)} \cdot \frac{u^{k-n/2}}{\Gamma(k-n/2+1)}, \tag{14}$$

where u is defined by (7)

2. μ odd and ν odd (n even)

$$R_{2k}(u) = \frac{1}{(-1)^{(\mu-1)/2}\pi^{(n-1)/2}2^{2k-1}\Gamma(k)} \cdot \frac{u^{k-n/2}}{\Gamma(k-n/2+1)} \tag{15}$$

if $k \geq n/2$ and

$$R_{2k}(u) = \frac{1}{(-1)^{(\mu-1)/2}\pi^{(n-1)/2}2^{2k-1}\Gamma(k)}\delta^{(n/2-k-1)} \tag{16}$$

if $k < n/2$.

On the other hand Aguirre and Trione in [6, p.123] prove the following formula

$$R_\alpha(u) * R_{2k}(u) = R_{\alpha+2k}(u) \tag{17}$$

for all μ and ν where the symbol $*$ stands for convolution and $\mu+\nu = n$ is the dimension of the space.

In this paper we find two formulae for two special convolution products

$$\delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u)$$

and

$$\delta^{(n/2-k-1)}(m^2+u) * \delta^{(n/2-l-1)}(m^2+u)$$

using the formulae (16), (17) and the following formula [10, p.123]

$$\delta^{(k-1)}(m^2+u) = \sum_{\nu=o}^{n/2-k-1} \frac{(m^2)^\nu}{\nu!}\delta^{(k+\nu-1)}(u) \tag{18}$$

which holds when n is even and $k < n/2-1$.

THEOREM 1. Let k and l be non-negative integers and n an even positive integer. Then

$$\delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u) = A_{k,l,n}\delta^{(n/2-k-l-1)}(u) \tag{19}$$

under conditions (i) μ and ν are both odd, and (ii) $0 \leq k+l \leq n/2-1$, where [5, p.148]

$$A_{k,l,n} = \frac{1}{2}(-1)^{(\mu-1)/2}\pi^{(n-2)/2}\frac{\Gamma(k)\Gamma(l)}{\Gamma(k+l)}, \tag{20}$$

and

$$\delta^{(n/2-k-1)}(u) = \frac{(n/2-k-1)!}{(-1)^{n/2-k-1}}\mathrm{res}_{\beta=-(n/2-k)}u^\beta \tag{21}$$

and u is defined by (7)

PROOF. From (16) and (17), we have,

$$\begin{aligned} & \delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u) \\ = \; & (-1)^{(\mu-1)/2}\pi^{(n-2)/2}2^{2k-1}\Gamma(k) \\ & \times\Gamma(l)(-1)^{(\mu-1)/2}\pi^{(n-2)/2}2^{2l-1}\left(R_{2k}(u) * R_{2l}(u)\right) \\ = \; & \Gamma(k)\Gamma(l)(-1)^{\mu-1}\pi^{n-2}2^{2(k+l)-2}R_{2(k+l)}(u) \end{aligned} \tag{22}$$

if μ and ν are both odd. Now from (22) and (16), we have

$$\delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u) = \frac{1}{2}(-1)^{(\mu-1)/2}\pi^{(n-2)/2}\frac{\Gamma(k)\Gamma(l)}{\Gamma(k+l)}\delta^{(n/2-k-l-1)}(u) \tag{23}$$

if μ and ν are both odd and $k+l<n/2$. From (23) we deduce (19). The proof is complete.

We remark that our new formula (19) is not a consequence of the convolution product

$$\delta^{(s)}(P_+)*\delta^{(t)}(P_+)=(-1)^{s+t}h_{s,t,n}P_-^{n/2-s-t-2} \tag{24}$$

which appear in [7], where

$$h_{s,t,n}=2^{-2}(-1)^{(\nu-1)/2}\pi^{(n+2)/2}\frac{\Gamma(n/2-s-1)\Gamma(n/2-t-1)}{\Gamma(n/2-s-t-1)\Gamma(n-s-t-2)} \tag{25}$$

and [11, p.278]

$$\delta^{(s)}(P_+)=(-1)^s s!\mathrm{res}_{\lambda=-s-1,s=0,1,2,\ldots}P_+^{\lambda} \tag{26}$$

and [11, p.254]

$$P_+^{\lambda},\varphi\rangle=\int_{P>0}(P(x))^{\lambda}\varphi(x)dx \tag{27}$$

and $P(x)=u(x)=u$ with u defined by (7).

In fact, putting $s=n/2-k-1$ and $t=n/2-l-1$ in (24) and (25), we have

$$\begin{aligned}&\delta^{(n/2-k-1)}(P_+)*\delta^{(n/2-l-1)}(P_+)\\ =\;&2^{-2}(-1)^{(\mu-1)/2}\pi^{(n-2)/2}\frac{\Gamma(k)\Gamma(l)(-1)^{n-k-l}}{\Gamma(k+l)\Gamma(k+l-n/2+1)}P_-^{k+l-n/2}\end{aligned} \tag{28}$$

provided $n/2\le k+l\le n-2$, where [11, p.269]

$$P_-^{\lambda},\varphi\rangle=\int_{P<0}(-P(x))^{\lambda}\varphi(x)dx. \tag{29}$$

Therefore our new convolution product formula (19) is in some way complementary to the formula (28).

THEOREM 2. Let k and l be non-negative integers and n an even positive integer. Then

$$\begin{aligned}&\delta^{(n/2-k-1)}(m^2+u)*\delta^{(n/2-l-1)}(m^2+u)\\ =\;&\delta^{(n/2-k-1)}(u)*\delta^{(n/2-l-1)}(u)+a_{\mu,n}\sum_{\nu=1}^{k-1}\frac{(m^2)^{\nu}}{\nu!}A_{k-\nu,l}\delta^{(n/2-k-l+\nu-1)}(u)\\ &+a_{\mu,n}\sum_{j=1}^{l-1}\frac{(m^2)^{j}}{j!}A_{k,l-j}\delta^{(n/2-k-l+j-1)}(u)+\\ &+a_{\mu,n}\sum_{\nu=1}^{k-1}\sum_{j=1}^{l-1}\frac{(m^2)^{\nu+j}}{\nu!}A_{k-\nu,l-j}\delta^{(n/2-k-l+\nu+j-1)}(u)\end{aligned} \tag{30}$$

under conditions *a*) μ and ν are both odd, and *b*) $0\le k+l\le n/2-1$, where

$$A_{s,t}=\frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)} \tag{31}$$

and

$$a_{\mu,n} = \frac{1}{2}(-1)^{(\mu-1)/2}\pi^{(n-1)/2}. \tag{32}$$

PROOF. From (18), we have

$$\begin{aligned} & \delta^{(n/2-k-1)}(m^2+u) * \delta^{(n/2-l-1)}(m^2+u) \\ = & \sum_{\nu=0}^{k-1}\sum_{j=0}^{l-1} \frac{(m^2)^{\nu+j}}{\nu!}\left\{A_{k-\nu,l-j}\delta^{(n/2-k-l+\nu+j-1)}(u)\right\}. \end{aligned} \tag{33}$$

Now using the formula (19), we have

$$\begin{aligned} & \delta^{(n/2-k-1)}(m^2+u) * \delta^{(n/2-l-1)}(m^2+u) \\ = & \sum_{\nu=0}^{k-1}\sum_{j=0}^{l-1} \frac{(m^2)^{\nu+j}}{\nu!}\left\{A_{k-\nu,l-j,n}\delta^{(n/2-k-l+\nu+j-1)}(u)\right\}. \end{aligned} \tag{34}$$

where $A_{k-\nu,l-j,n}$ is defined by (20). From (34) and (19), we obtain

$$\begin{aligned} & \delta^{(n/2-k-1)}(m^2+u) * \delta^{(n/2-l-1)}(m^2+u) \\ = & \delta^{(n/2-k-1)}(u) * \delta^{(n/2-l-1)}(u) + +a_{\mu,n}\sum_{\nu=1}^{k-1}\frac{(m^2)^{\nu}}{\nu!}A_{k-\nu,l}\delta^{(n/2-k-l+\nu-1)}(u) \\ & +a_{\mu,n}\sum_{j=1}^{l-1}\frac{(m^2)^{j}}{j!}A_{k,l-j}\delta^{(n/2-k-l+j-1)}(u) \\ & a_{\mu,n}\sum_{\nu=1}^{k-1}\sum_{j=1}^{l-1}\frac{(m^2)^{\nu+j}}{\nu!j!}A_{k-\nu,l-j}\delta^{(n/2-k-l+\nu+j-1)}(u) \end{aligned} \tag{35}$$

where $a_{\mu,n}$ is defined by the (32). The formula (35) coincides with (30).

It's clear that putting $m^2 = 0$ in (30) we obtain (19).

We remark that (30) is not a consequence of the equality

$$\begin{aligned} & \delta^{(s-1)}(m^2+P) * \delta^{(t-1)}(m^2+P) \\ = & \delta^{(s-1)}(P_+) * \delta^{(t-1)}(P_+) + \sum_{r=1}^{n/2-s-t}\frac{(m^2)^r}{r!}C_{n/2-s-r,n/2-t-r,r,l}P_-^{n/2-k-t-r} \end{aligned} \tag{36}$$

which appear in [8], where

$$\begin{aligned} & C_{n/2-s-r,n/2-t-r,r,l} \\ = & \pi^{n/2+1}2^{-2}(-1)^{(\nu-1)/2}\frac{\Gamma(n/2-s-r)\Gamma(n/2-t-r)}{\Gamma(n/2-s-t+1)\Gamma(n-s-r-t-r)}. \end{aligned} \tag{37}$$

In fact, putting $s = n/2-k-1$ and $t = n/2-l-1$ in (36) and (37), we have

$$\begin{aligned} & \delta^{(n/2-k-1)}(m^2+P) * \delta^{(n/2-l-1)}(m^2+P) \\ = & \delta^{(n/2-k-1)}(P_+) * \delta^{(n/2-l-1)}(P_+) + \sum_{r=1}^{k+l-n/2}\frac{(m^2)^r}{r!}C_{k-r,l-r,r,l}P_-^{k+l-n/2-r} \end{aligned}$$

under conditions μ and ν odd and $k+l \geq n/2+r$, where

$$C_{k-r,l-r,r,l} = \pi^{n/2+1} 2^{-2} (-1)^{(\nu-1)/2} \frac{\Gamma(k-r)\Gamma(l-r)}{\Gamma(k+l-n/2)\Gamma(k+l-2r)}.$$

Therefore (30) is in some way complementary to (36).

References

[1] Y. Nozaki, On Riemann-Liouville integral of ultra-hyperbolic type, Kodai Mathematical Seminar Reports, 6(2)(1964), 69-87.

[2] A. Erdelyi, Ed. Higher Trascendental Functions, Vol. I, McGraw-Hill, New York, 1953.

[3] M. Riesz, L'intégrale de Riemann-Liouville et le probléme de Cauchy pour l'équation des ondes, Comm. Sém. Math. Univ. de Lund, 4(1939), 28-42.

[4] S. E. Trione, On Marcel Riesz's ultrahyperbolic kernel, preprint.

[5] M. A. Aguirre T., The distributional Hankel transform of Marcel Riesz's ultrahyperbolic kernel, Studies in Applied Mathematics 93(1994), 133-162.

[6] M. A. Aguirre T. and S. E. Trione, The distributional convolution products of Marcel Riesz's ultra-hyperbolic kernel, Revista de la Unión Matemática Argentina, 39(1995), 115-124.

[7] M. A. Aguirre T., The distributional convolution product of k-th derivative of Dirac delta in hypercone, preprint.

[8] M. A. Aguirre T., Convolution product of $(k-1)$-th derivative of Dirac's delta in m^2+P, preprint.

[9] M. A. Aguirre T., The expansion of $\delta^{(k-1)}(m^2+P)$, Integral Transform and Special Functions, 8(1-2)(1999), 139-148..

[10] M. A. Aguirre T., The expansion and Fourier's transform of $\delta^{(k-1)}(m^2+P)$, Integral Transform and Special Functions, 3(2)(1995), 113-134.

[11] I. M.Gelfand and G. E. Shilov., Generalized Function, Vol. I, Academic Press, New York, 1964.