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Periodic Solutions of Abstract Difference Equations *

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Abstract

We investigate whether a control mechanism can be introduced to support a periodic solution for an abstract difference equation modeling a diffusion problem. An existence theorem is proved and estimates of the norms of the periodic solution is also obtained.

1 Introduction

To motivate what follows, consider a large number of equally divided cells contained in a "very long" straight tube, each of which contains a certain solute dissolved in a unit volume of solvent. These cells are separated from each other by semipermeable membranes through which the solute may flow but not the solvent. Let us denote by $u_i^{(t)}$ the amount of solute dissolved in the i-th cell and at different time periods $t = 0, 1, 2, ...$. Since each cell contains a unit volume of solvent, $u_i^{(t)}$ also represents the concentration of solute in the i-th cell. During the time period t, if the concentration $u_{i-1}^{(t)}$ is higher than $u_i^{(t)}$, solute will flow from the $(i-1)$-th cell to the i-th cell. The amount of increase is $u_i^{(t+1)} - u_i^{(t)}$, and it is reasonable to postulate that the increase is proportional to the difference $u_{i-1}^{(t)} - u_i^{(t)}$. Similarly, solute will flow from the $(i+1)$-th cell to the i-th cell if $u_{i+1}^{(t)} > u_i^{(t)}$. Thus, it is reasonable the total effect is

$$u_i^{(t+1)} - u_i^{(t)} = \gamma \left(u_{i-1}^{(t)} - u_i^{(t)}\right) + \gamma \left(u_{i+1}^{(t)} - u_i^{(t)}\right),$$

where γ is a proportionality constant. If the permeability of the membrane is time dependent, and if a time dependent control is introduced, it is plausible that the governing equation becomes

$$u_i^{(t+1)} - u_i^{(t)} = \gamma_t \left(u_{i+1}^{(t)} - 2u_i^{(t)} + u_{i-1}^{(t)}\right) + G\left(t, u_i^{(t)}\right).$$

By writing $\left\{u_i^{(t)}\right\}_{i=-\infty}^{\infty}$ as x_t, we may write the above equation in the form

$$x_{t+1} - x_t = \gamma_t J x_t + g_t(x_t),\ t = 0, 1, 2, ...,$$

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where J is a doubly infinite matrix with diagonal elements equal to -2, and superdiagonal as well as subdiagonal elements equal to 1.

Existence of periodic solutions to difference equations similar to the one derived above have been studied by several authors, see e.g. [1-5]. Here, an important question arises naturally as to whether the control mechanism can maintain a periodic solution for the above equation. To this end, let C be the set of complex numbers and let X be a complex Banach space with a norm $\|\cdot\|$. We will denote the identity operator defined on X by I and denote the closed ball with radius r by $-_r$, i.e. $-_r = \{v \in X| \ \|v\| \le r\}$, where $0 < r \le \infty$. Consider sequences of the form $\{x_k\}_{k=0}^{\infty}$ in X which satisfies the perturbed difference equation

$$x_{k+1} = A_k x_k + F_k(x_k), \ k = 0, 1, 2, ..., \tag{1}$$

where $\{A_k\}_{k=0}^{\infty}$ is a periodic sequence of bounded operators defined on X such that

$$A_k = A_{k+T}, \ k \ge 0, \tag{2}$$

and

$$I - A_0 A_1 \cdots A_{T-1} \text{ is invertible}, \tag{3}$$

and $\{F_k\}_{k=0}^{\infty}$ is a periodic sequence of functions from $-_r$ into X such that

$$F_k = F_{k+T}, \ k \ge 0, \tag{4}$$

and

$$\|F_k(x) - F_k(y)\| \le q \|x - y\|, \ k = 0, 1, ..., T-1; \ x, y \in -_r; \ q > 0. \tag{5}$$

In the special case when $X = C^n$, $A_k = A$ for $k \ge 0$ and $\|F_k(x)\| \le \alpha \|x\| + \beta$ for $k \ge 0$ and $x \in X$, equation (1) has been studied [6], and a periodic solution is found under suitable conditions on A, α, and β. Here, we will also be interested with the existence of periodic solutions of our general abstract difference equation (1), as well as estimates of their norms.

2 Main Results

To accomplish our goals, let us set

$$U(k, j) = \prod_{i=j}^{k-1} A_i, \ 0 \le j < k \le T,$$

and set $U(j, j) = I$ for $j \ge 0$. Then it is easily checked that the unique solution of the equation

$$y_{k+1} = A_k y_k + f_k, \ f_k \in X, \ k = 0, 1, ...,$$

is given by

$$y_k = U(k, 0) y_0 + \sum_{j=0}^{k-1} U(k, j) f_j, \ k = 1, 2, ...$$

Thus the periodic boundary value problem

$$y_{k+1} = A_k y_k + f_k,\ f_k \in X,\ k = 0, 1, ..., T,$$

$$y_0 = y_T$$

has a solution provided

$$y_0 = y_T = U(T, 0) y_0 + \sum_{j=0}^{T-1} U(T, j) f_j,$$

or

$$y_0 = (I - U(T, 0))^{-1} \sum_{j=0}^{T-1} U(T, j) f_j,$$

and in such a case, this solution is given by

$$y_k = U(k, 0) \left(I - U(T, 0)\right)^{-1} \sum_{j=0}^{T-1} U(T, j) f_j + \sum_{j=0}^{k-1} U(k, j) f_j,\ 0 \le k \le T, \tag{6}$$

and its maximum norm satisfies

$$\max_{0\le k\le T-1} \|y_k\| \le \gamma_T \max_{0\le k\le T-1} \|f_k\|, \tag{7}$$

where

$$\gamma_T = \max_{0\le k\le T-1} \sum_{j=0}^{T-1} \left\{ \left\| U(k, 0)(I - U(T, 0))^{-1} U(T, j) \right\| + \|U(k, j)\| \right\}. \tag{8}$$

THEOREM 1. Under the conditions (2)-(5), if $\gamma_T(qr + l_T) < r < \infty$, where $l_T = \max_{0\le k\le T-1} \|F_k(0)\|$, then the periodic boundary value problem

$$x_{k+1} = A_k x_k + F_k(x_k),\ k = 0, 1, ..., T-1, \tag{9}$$

$$x_0 = x_T, \tag{10}$$

has a unique solution.

PROOF. Let Φ be the Cartesian product X^T. When equipped with the maximum norm defined by

$$\|(x_0, ..., x_{T-1})\|_\Phi = \max_{0\le k\le T-1} \|x_k\|,\ (x_0, ..., x_{T-1}) \in \Phi,$$

Φ becomes a Banach space. Let

$$G = \{x \in \Phi |\ \|x\|_\Phi \le r\}.$$

Furthermore, let $\Psi : G \to \Phi$ be defined as follows: for each $x = (x_0, ..., x_{T-1}) \in G$, define $(\Psi x)_0 = 0$ and

$$(\Psi x)_k = U(k,0)\,(I - U(T,0))^{-1} \sum_{j=0}^{T-1} U(T,j)F_j(x_j) + \sum_{j=0}^{k-1} U(k,j)F_j(x_j)$$

for $0 \le k \le T-1$. Then for each $x \in G$,

$$\|\Psi x\|_\Phi \le \gamma_T \max_{0\le k\le T-1} \|F_k(x_k)\| \le \gamma_T \max_{0\le k\le T-1} \{q\,\|x_k\| + l_T\} \le \gamma_T\,(qr + l_T) < r,$$

which implies $\Psi x \in G$. Furthermore,

$$\|\Psi x - \Psi y\|_\Phi \le \gamma_T q\,\|x - y\|_\Phi\,,\ x, y \in G.$$

Since $\gamma_T(qr + l_T) < r$ implies $0 \le \gamma_T l_T < r(1 - \gamma_T q)$, we see that Ψ is a contraction mapping on G. By Banach's fixed point theorem, Ψ has a unique fixed point u in G. It is easily seen that u is a solution of (9)-(10). The proof is complete.

We remark that the unique solution asserted in the above theorem satisfies

$$\max_{0\le k\le T-1} \|x_k\| \le \frac{\gamma_T l_T}{1 - q\gamma_T}. \tag{11}$$

Indeed, if $\{x_k\}_{k=0}^T$ is such a solution, then in view of (7), (8) and (5), we will have

$$\max_{0\le k\le T-1} \|x_k\| \le \gamma_T \max_{0\le k\le T-1} \|F_k(x_k)\| \max_{0\le k\le T-1} \|x_k\| \le \gamma_T \max_{0\le k\le T-1} \{q\,\|x_k\| + l_T\}\,,$$

which implies (11).

There are at least two important variations of Theorem 1. First of all, if $r = \infty$, we may replace the assumption $\gamma_T(qr + l_T) < r < \infty$ by $\gamma_T l_T < 1$ in the above theorem: Under the conditions (2)-(5) where $r = \infty$, if $\gamma_T l_T < 1$, then the periodic boundary problem (9)-(10) has a unique solution $\{x_k\}_{k=0}^T$ which satisfies (11). Second, if we assume that $\gamma_T l_T > 0$, then the condition $\gamma_T(qr + l_T) < r$ in the above Theorem can be replaced by the condition $\gamma_T(qr + l_T) \le r$: Under the conditions (2)-(5) and $\gamma_T l_T > 0$, if $\gamma_T(qr + l_T) \le r$, where $l_T = \max_{1\le k\le T-1} \|F_k(0)\|$, then the periodic boundary value problem (9)-(10) has a unique solution $\{x_k\}_{k=0}^T$ which satisfies (11). The proofs of these statements are not much different from that of Theorem 1 and hence omitted.

The constant γ_T defined by (8) is difficult to evaluate. To simplify matters, let us set $Q_{j,j} = 1$ for $j \ge 0$,

$$Q_{k,j} = \prod_{i=j}^{k-1} \|A_i\|\,,\ 0 \le j < k \le T,$$

and

$$Q_{k,j} = \frac{1}{Q_{j,k}},\ 0 \le k < j \le T.$$

Then $\|U(k,j)\| \le Q_{k,j}$ for $0 \le k \le j \le T-1$ and $Q_{k,j}Q_{j,i} = Q_{k,i}$ for $i \le j \le k$. Furthermore, if $Q_{T,0} < 1$, then

$$\begin{aligned}
\gamma_T &\le \max_{0\le k\le T-1} \sum_{j=0}^{T-1} \left\{ Q_{k,0}(1-Q_{T,0})^{-1}Q_{T,j} + Q_{k,j} \right\} \\
&\le \max_{0\le k\le T-1} \sum_{j=0}^{T-1} Q_{k,0} \left\{ (1-Q_{T,0})^{-1}Q_{T,0} + 1 \right\} Q_{0,j} \\
&\le \frac{1}{1-Q_{T,0}} \max_{0\le k\le T-1} Q_{k,0} \sum_{j=0}^{T-1} \frac{1}{Q_{j,0}}.
\end{aligned}$$

THEOREM 2. Assume that $Q_{T,0} < 1$. Let $l_T = \max_{1\le k\le T-1} \|F_k(0)\|$, and

$$\rho_T = \frac{1}{1-Q_{T,0}} \max_{0\le k\le T-1} Q_{k,0} \sum_{j=0}^{T-1} \frac{1}{Q_{j,0}}.$$

Under the conditions (2)-(5), if $\rho_T(qr + l_T) < r$, then the boundary value problem (9)-(10) has a unique solution $\{u_k\}_{k=0}^T$. Moreover, the inequality

$$\max_{0\le k\le T-1} \|u_k\| \le \frac{\rho_T l_T}{1-q\rho_T}$$

is valid.

The proof is similar to that of Theorem 1 and thus omitted.

3 An Example

We now turn to our diffusion problem. Let us find a control such that our problem can be written in the form

$$u_i^{(t+1)} - u_i^{(t)} = a_t u_{i+1}^{(t)} - b_t u_i^{(t)} + c_t u_{i-1}^{(t)} + G\left(t, u_i^{(t)}\right),$$

where $i = 0, \pm 1, \ldots$, and $t = 0, 1, \ldots$. The above partial difference equation can be written in the form

$$u_{t+1} = A_t u_t + F_t(u_t),\ t = 0, 1, 2, \ldots,$$

where A_t is a doubly infinite matrix with diagonal elements equal to $1 + b_t$, and superdiagonal and subdiagonal elements equal to a_t and c_t, respectively. Suppose $\{a_t\}$, $\{b_t\}$ and $\{c_t\}$ are T-periodic real sequences, then the matrix sequence $\{A_t\}$ is also T-periodic. Take X to be the space of doubly infinite bounded sequences endowed with the supremum norm. Then

$$\|A_t\| = |1 - b_t| + a_t + c_t.$$

Therefore,

$$Q_{k,j} = \prod_{i=j}^{k-1} \left(|1 - b_i| + a_i + c_i \right).$$

Suppose further that $Q_{T,0} < 1$ (which is satisfied, for example, when $a_t + c_t < b_t < 1$ for all t.) Then the quantity ρ_T can be calculated in a straightforward manner, and Theorem 2 can then be applied.

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